# SRINIVASAN ENGINEERING COLLEGE


***A Unit of Dhanalakshmi Srinivasan Group of Institutions***

*(Approved by AICTE, New Delhi & Affiliated to Anna University, Chennai)*

PERAMBALUR – 621 212.


## *DEPARTMENT OF AERONAUTICAL ENGINEERING*

# *AE 2402 Computational Fluid Dynamics*

Class Notes

**Name** :……………………………………………………………….

**Register Number** :……………………………………………………………….

**Semester** :……………………………………………………………….

**Academic Year** :……………………………………………………………….

| AE2402 | COMPUTATIONAL FLUID DYNAMICS | L | T | P | C |
|---|---|---|---|---|---|
| | | 3 | 0 | 0 | 3 |

**OBJECTIVE**
To study the flow of dynamic fluids by computational methods

**UNIT I FUNDAMENTAL CONCEPTS 10**
Introduction - Basic Equations of Fluid Dynamics - Incompressible In viscid flows: Source, vortex and doublet panel, methods - lifting flows over arbitrary bodies. Mathematical properties of Fluid Dynamics Equations -_ Elliptic, Parabolic and Hyperbolic equations - Well posed problems - discretization of partial Differential Equations. Explicit finite difference methods of subsonic, supersonic and viscous flows.

**UNIT II GRID GENERATION 7**
Structured grids. Types and transformations. Generation of structured grids. Unstructured grids. Delany triangulation.

**UNIT III DISCRETIZATION 8**
Boundary layer Equations and methods of solution -Implicit time dependent methods for inviscid and viscous compressible flows - Concept of numerical dissipation –Stability properties of explicit and implicit methods - Conservative upwind discretization for Hyperbolic systems - Further advantages of upwind differencing.

**UNIT IV FINITE ELEMENT TECHNIQUES 6**
Overview of Finite Element Techniques in Computational Fluid Dynamics. Strong and Weak Formulations of a Boundary Value Problem.

**UNIT V FINITE VOLUME TECHNIQUES 14**
Finite Volume Techniques - Cell Centered Formulation - Lax - Vendoroff Time Stepping - Runge - Kutta Time Stepping - Multi - stage Time Stepping - Accuracy -. Cell Vertex Formulation - Multistage Time Stepping - FDM -like Finite Volume Techniques – Central and Up-wind Type Discretizations - Treatment of Derivatives. Flux – splitting schemes. Pressure correction solvers – SIMPLE, PESO. Vorticity transport formulation. Implicit/semi-implicit schemes.

**TOTAL: 45 PERIODS**

**TEXT BOOK**
1. Fletcher, C.A.J., “Computational Techniques for Fluid Dynamics”, Vols. I and II, Springer - Verlag, Berlin, 1988.

**REFERENCES**
1. John F. Wendt (Editor), “Computational Fluid Dynamics - An Introduction”, Springer – Verlag, Berlin, 1992.
2. Charles Hirsch, “Numerical Computation of Internal and External Flows”, Vols. I and II. John Wiley & Sons, New York, 1988.
3. Klaus A Hoffmann and Steve T. Chiang. “Computational Fluid Dynamics for Engineers”, Vols. I & II Engineering Education System, P.O. Box 20078, W. Wichita, K.S., 67208 - 1078 USA, 1993.
4. Anderson, Jr.D., “Fundamentals of Aerodynamics”, McGraw-Hill, 2000.

# COMPUTATIONAL FLUID DYNAMICS



## Introduction

It needs both a high speed computer and efficient computational methods. It is mainly composed of three elements

i) Aerodynamics theory (Fluid dynamics)
ii) Applied mathematics
iii) Advanced computers. (Super computers).

## Basic Fluid Dynamic Equations

The equation of continuity, momentum and energy are the Basic fluid dynamic equations. The equations are written in cartesian coordinate system, and the underlying assumptions are

i) The gas is a single component gas.
ii) There is no volume force acting on it.
iii) The gas is not radiating.

The three fundamental physical principles upon which all of fluid dynamics is based are

i) Mass is conserved.
ii) Newton's second law, $F = ma$
iii) Energy is conserved.

The fluid flow equations that obtained directly by applying the fundamental physical principles to a finite control volume are in integral form.

These integral forms of the governing equations can be manipulated to indirectly obtain partial differential equations.

The equations so obtained from the finite control volume fixed in space, in either integral or partial differential form, are called the conservation form of the governing equations.

The equations obtained from the finite control volume moving with the fluid, in either integral or partial differential form, are called the non-conservation form of the governing equations.

## The Substantial Derivative

An infinitesimally small fluid element moving with the flow. The motion of this fluid element is shown in more detail in following figure. Here, the fluid element is moving through cartesian space.

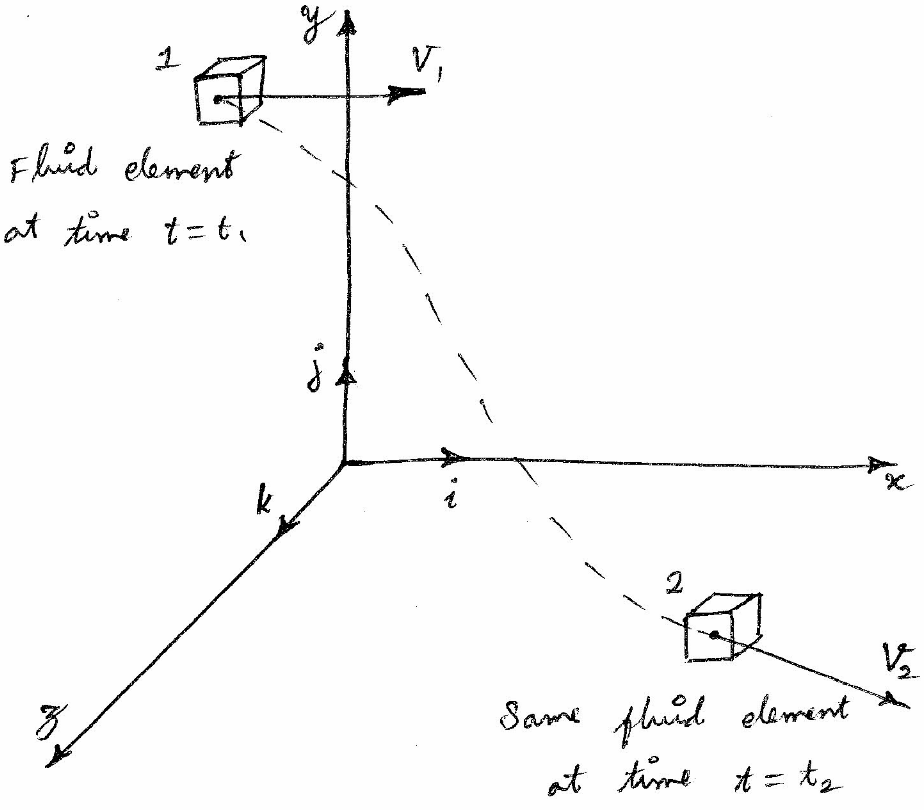



Illustration for the Substantial derivative

The unit vectors along the x, y and z axes are i, j and k respectively. The vector velocity field in this cartesian space is given by

$$\vec{V} = u\vec{i} + v\vec{j} + w\vec{k}.$$

where, the x, y and z components of velocity are given, respectively, by

$$u = u(x, y, z, t)$$

$$v = v(x, y, z, t)$$

$$w = w(x, y, z, t)$$

We are considering in general an <u>unsteady flow</u>, where u, v and w are functions of both space and time t. In addition, the scalar density field is given by

$$\rho = \rho(x, y, z, t)$$

At time $t_1$, the fluid element is located at point 1. Then,

$$\rho_1 = \rho(x_1, y_1, z_1, t_1)$$

At a later time $t_2$, the same fluid element has moved to point 2. Hence,

$$\rho_2 = \rho(x_2, y_2, z_2, t_2)$$

Since $\rho = \rho(x, y, z, t)$ we can expand this function in a Taylor series about point 1.

$$\rho_2 = \rho_1 + \left(\frac{\partial \rho}{\partial x}\right)_1 (x_2 - x_1) + \left(\frac{\partial \rho}{\partial y}\right)_1 (y_2 - y_1) + \left(\frac{\partial \rho}{\partial z}\right)_1 (z_2 - z_1)$$

$$+ \left(\frac{\partial \rho}{\partial t}\right)_1 (t_2 - t_1) + (\text{higher-order terms})$$

Dividing by $(t_2 - t_1)$ and ignoring higher-order terms, we obtain

$$\frac{\rho_2 - \rho_1}{t_2 - t_1} = \left(\frac{\partial \rho}{\partial x}\right)_1 \frac{x_2 - x_1}{t_2 - t_1} + \left(\frac{\partial \rho}{\partial y}\right)_1 \frac{y_2 - y_1}{t_2 - t_1} + \left(\frac{\partial \rho}{\partial z}\right)_1 \frac{z_2 - z_1}{t_2 - t_1} + \left(\frac{\partial \rho}{\partial t}\right)_1$$

Examine the left side of the above eqn. This is Physically the average time rate of change in density of the fluid element as it moves from point 1 to 2. In the limit, as $t_2$ approaches $t_1$, this term becomes

$$\lim_{t_2 \to t_1} \frac{\rho_2 - \rho_1}{t_2 - t_1} \equiv \frac{D\rho}{Dt}$$



And the right side terms are.

$$\lim_{t_2 \to t_1} \frac{x_2 - x_1}{t_2 - t_1} \equiv u$$

$$\lim_{t_2 \to t_1} \frac{y_2 - y_1}{t_2 - t_1} \equiv v$$

$$\lim_{t_2 - t_1} \frac{z_2 - z_1}{t_2 - t_1} \equiv w$$

Then the eqn is

$$\frac{D\rho}{Dt} = u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} + \frac{\partial \rho}{\partial t}$$

So, the substantial derivative in cartesian coordinates.

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}.$$

W.K.T in cartesian coordinates, the vector operator $\nabla$ is defined as,

$$\nabla \equiv \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}.$$

So, the substantial derivative can be written as,

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + (\vec{V}.\nabla)$$

The above eqn represents a definition of the substantial derivative operator in vector notation. Thus, it is valid for any coordinate system. Where,

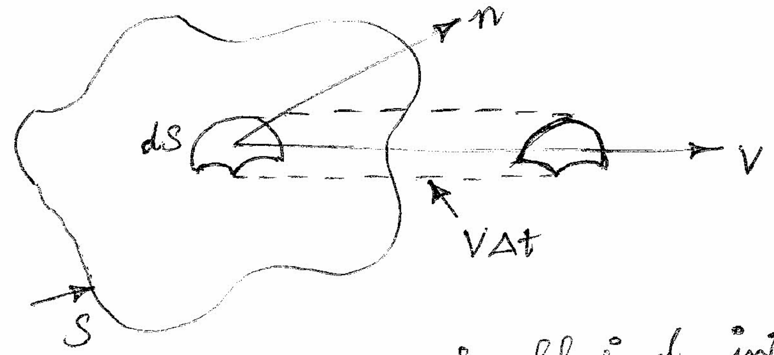
$\frac{D}{Dt}$ - Substantial derivative, which is physically the time rate of change following a moving fluid element.

$\frac{\partial}{\partial t}$ - Local derivative, which is physically the time rate of change at a fixed point.

$\vec{V}.\nabla$ - convective derivative, which is physically the time rate of change due to the movement of the fluid element.

## Physical Meaning of $\nabla \cdot \vec{V}$

Consider a control volume moving with the fluid. This control volume is always made ups of the same fluid particles as it moves with the flow. So, its mass is fixed, invariant with time. However, its volume $\mathcal{V}$ and control surface $S$ are changing with time as it moves to different regions of the flow where different values of $\rho$ exist. So, this moving control volume of fixed mass is constantly increasing or decreasing its volume and is changing its shape, depending on the characteristics of the flow.




Moving control volume used for the physical interpretation of the divergence of velocity.

Consider an infinitesimal element of the surface $dS$ moving at the local velocity $\vec{V}$. The change in the volume of the control volume $\Delta\mathcal{V}$, due to just the movement of $dS$ over a time increment $\Delta t$ is equal to the volume of the long, thin cylinder with base area $dS$ and altitude $(\vec{V}\Delta t)\cdot\vec{n}$. where

$\vec{n} \rightarrow$ unit vector perpendicular to the surface at $dS$

$$\Delta \mathcal{V} = \left[(\vec{V}\Delta t).\vec{n}\right] dS$$
$$= (\vec{V}\Delta t).d\vec{S}$$

Over the time increment $\Delta t$, the total change in volume of the whole control volume is equal to the summation over the total control surface.

In the limit as $dS \to 0$, the sum becomes the surface integral.

$$\oiint_S (\vec{V}\Delta t).\, d\vec{S}$$

If this integral is divided by $\Delta t$, the result is physically the time rate of change of the control volume

$$\frac{D\mathcal{V}}{Dt} = \oiint_S \vec{V}.d\vec{S}$$

Applying the divergence theorem to right side, we obtain

$$\frac{D\mathcal{V}}{Dt} = \oiiint_V (\nabla.\vec{V})\, d\mathcal{V}$$

Assume that $\delta\mathcal{V}$ is small enough such that $\nabla.\vec{V}$ is essentially the same value throughout $\delta\mathcal{V}$.

$$\frac{D(\delta \mathcal{V})}{Dt} = (\nabla \cdot \vec{V})\, \delta \mathcal{V}$$

$$\boxed{\nabla \cdot \vec{V} = \frac{1}{\delta \mathcal{V}} \frac{D(\delta \mathcal{V})}{Dt}}$$

So, $\nabla \cdot \vec{V}$ is physically the time rate of change of the volume of a moving fluid element, per unit volume.

## The continuity Equation.

"Mass is conserved," is the physical principle of the continuity eqn.

Consider the model of a moving fluid element. The mass of this element is fixed, and is given by $\delta m$. The volume of this element is $\delta \mathcal{V}$.

$$\delta m = \rho\, \delta \mathcal{V}$$

Since mass is conserved, we can state that the time-rate-of-change of the mass of the fluid element is zero as the element moves along with the flow.

$$\frac{D(\delta m)}{Dt} = 0$$

Then,

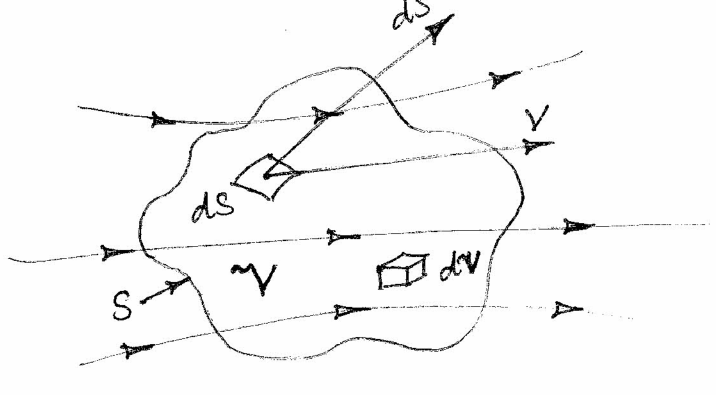
$$\frac{D(\rho\,\delta \mathcal{V})}{Dt} = \delta\mathcal{V}\frac{D\rho}{Dt} + \rho\frac{D(\delta\mathcal{V})}{Dt} = 0$$

$$\frac{D\rho}{Dt} + \rho\left[\frac{1}{\delta\mathcal{V}}\frac{D(\delta\mathcal{V})}{Dt}\right] = 0$$

$$\boxed{\frac{D\rho}{Dt} + \rho\,\nabla\cdot\vec{V} = 0}$$

The above eqn is the continuity eqn in non-conservative form.

Now consider the model of a finite control volume fixed in space.




Finite control volume fixed in space.

At a point on the control surface, the flow velocity is $\vec{V}$ and the vector elemental surface area is $d\vec{S}$. Also let $d\mathcal{V}$ be an elemental volume inside the finite control volume. For the physical principle.

$$(B)\left\{\begin{array}{l}\text{Net mass flow out}\\ \text{of control volume}\\ \text{through surface S}\end{array}\right\} = \left\{\begin{array}{l}\text{time rate of decrease}\\ \text{of mass inside}\\ \text{control volume}\end{array}\right\}(C)$$

The mass flow of a moving fluid across any fixed surface is

$$\dot{m} = \rho \vec{V} \cdot d\vec{S}$$

The net mass flow out of the entire control volume through the control surface S is the summation over S of the elemental mass flows. So,

$$B = \oiint_S \rho \vec{V} \cdot d\vec{S}$$

The mass contained within the elemental volume $d\mathcal{V}$ is $\rho\, d\mathcal{V}$. The total mass inside the control volume is

$$\iiint_{\mathcal{V}} \rho\, d\mathcal{V}$$

The time rate of decrease of mass inside control volume $\mathcal{V}$

$$C = -\frac{\partial}{\partial t} \iiint_{\mathcal{V}} \rho\, d\mathcal{V}$$

$$\therefore \quad \oiint_S \rho \vec{V} \cdot d\vec{S} = -\frac{\partial}{\partial t} \iiint_{\mathcal{V}} \rho\, d\mathcal{V}$$

$$\boxed{\frac{\partial}{\partial t} \iiint_{\mathcal{V}} \rho\, d\mathcal{V} + \oiint_S \rho \vec{V} \cdot d\vec{S} = 0}$$



This conservation form is the integral form of the continuity equation.

Let us cast the above eqn in the form of a differential equation. Since the control volume is fixed in space, the limits of integration for the integrals are constant, and hence the time derivative $\frac{\partial}{\partial t}$ can be placed inside the integral.

$$\iiint_{\mathcal{V}} \frac{\partial \rho}{\partial t} \, d\mathcal{V} + \oiint_S \rho \vec{V} \cdot d\vec{S} = 0$$

Applying divergence theorem, the surface integral can be expressed as a volume integral.

$$\therefore \quad \oiint_S (\rho \vec{V}) \cdot d\vec{S} = \iiint_{\mathcal{V}} \nabla \cdot (\rho \vec{V}) \, d\mathcal{V}$$

Then,

$$\iiint_{\mathcal{V}} \frac{\partial \rho}{\partial t} \, d\mathcal{V} + \iiint_{\mathcal{V}} \nabla \cdot (\rho \vec{V}) \, d\mathcal{V} = 0$$

$$\iiint_{\mathcal{V}} \left[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) \right] d\mathcal{V} = 0$$

Since the finite control volume is arbitrarily drawn in space, the only way for the integral to equal zero is for the integrand to be zero at every point within the control volume.

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0}$$

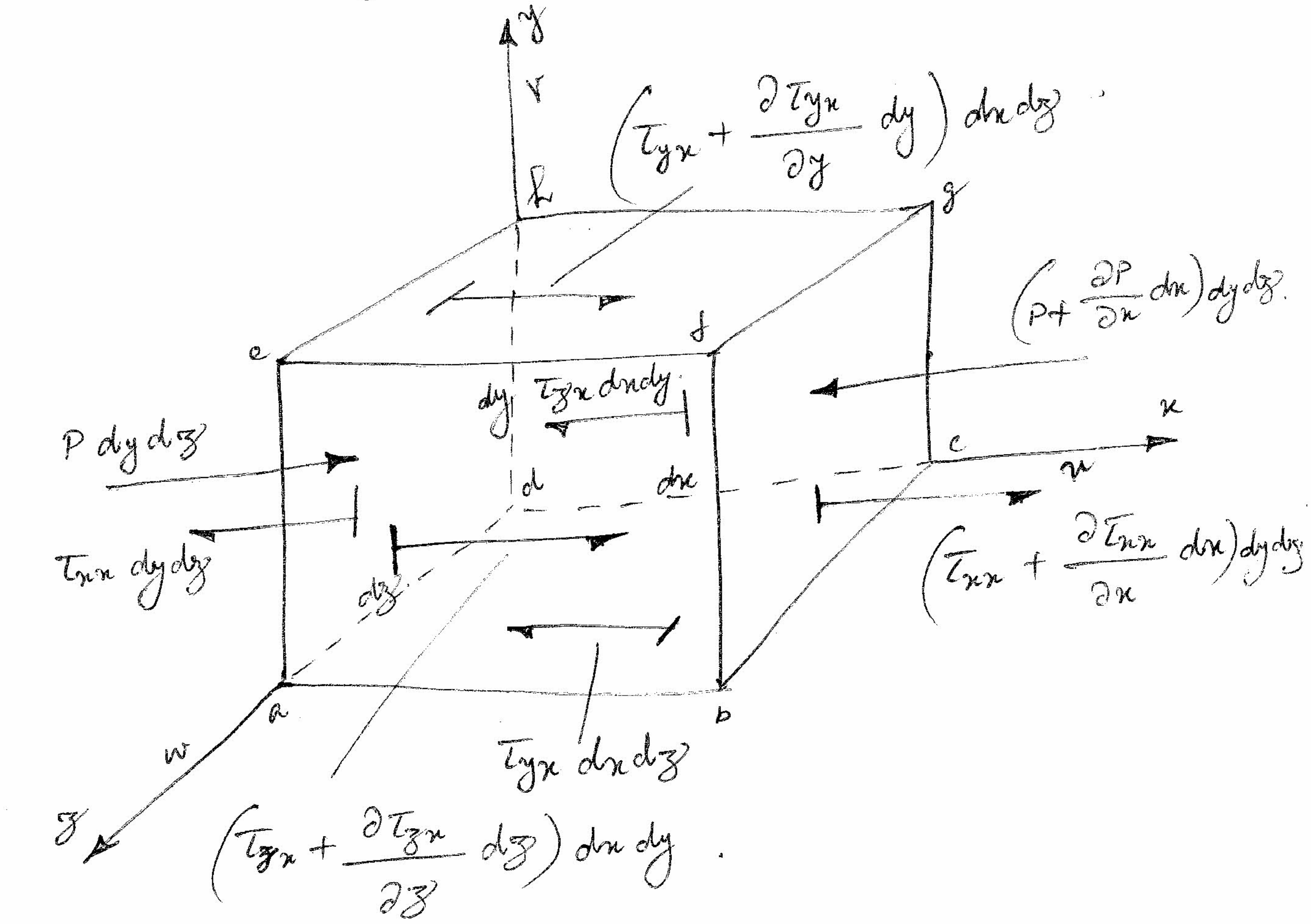
This conservation form is the partial differential form of the continuity equation.

## The Momentum Equation

The Newton's II law is the fundamental physical principle of the momentum equation.

$$\vec{F} = m\vec{a}$$

The net force on the fluid element equals its mass times the acceleration of the element. Let us consider a moving fluid element with only the x-component.



Then,

$$F_x = m a_x$$

The force experienced by the moving fluid element in $x$- direction.

i) Body force $\rightarrow$ gravitational, electric & magnetic forces.

ii) Surface force $\rightarrow$ a) Pressure

b) Shear & normal stress.

$$\left\{\begin{array}{c}\text{Body force on the} \\ \text{fluid element acting} \\ \text{in the } x\text{- direction}\end{array}\right\} = \rho f_x \,(dx\,dy\,dz)$$

$$\left\{\begin{array}{c}\text{Net surface force in the} \\ x\text{- direction}\end{array}\right\} = \left(\cancel{p} - \cancel{p} - \frac{\partial p}{\partial x} dx\right) dy\,dz$$

$$+ \left(\cancel{\tau_{xx}} + \frac{\partial \tau_{xx}}{\partial x} dx - \cancel{\tau_{xx}}\right) dy\,dz$$

$$+ \left(\cancel{\tau_{yx}} + \frac{\partial \tau_{yx}}{\partial y} dy - \cancel{\tau_{yx}}\right) dx\,dz$$

$$+ \left(\cancel{\tau_{zx}} + \frac{\partial \tau_{zx}}{\partial z} dz - \cancel{\tau_{zx}}\right) dx\,dy$$

$$= \left(-\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)$$

$$\times\, dx\,dy\,dz.$$

Then the total force in the x-direction.

$$F_x = \left(\rho f_x - \frac{\partial P}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) dx\, dy\, dz.$$

For right hand side of Newton's II law.

$$ma = \rho\, dx\, dy\, dz \times \frac{Du}{Dt}$$

Then.

$$\rho \frac{Du}{Dt} \cancel{dx\,dy\,dz} = \left(\rho f_x - \frac{\partial P}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) \cancel{dx\,dy\,dz}$$

$$\boxed{\rho \frac{Du}{Dt} = \rho f_x - \frac{\partial P}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}}$$

Above eqn is the x-component of the momentum eqn for a viscous flow.

IIIly.

$$\boxed{\rho \frac{Dv}{Dt} = \rho f_y - \frac{\partial P}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}}$$

$$\boxed{\rho \frac{Dw}{Dt} = \rho f_z - \frac{\partial P}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}.}$$

The above equations are momentum equation in Non-conservation form and is called Navier-Stokes equations.

The Navier-stokes equations can be obtained in conservation form as follows.

From the definition of the substantial derivative.

$$\rho \frac{Du}{Dt} = \rho \frac{\partial u}{\partial t} + \rho \vec{V} \cdot \nabla u$$

Also, expanding the following derivative

$$\frac{\partial (\rho u)}{\partial t} = \rho \frac{\partial u}{\partial t} + u \frac{\partial \rho}{\partial t}$$

Then,

$$\rho \frac{\partial u}{\partial t} = \frac{\partial (\rho u)}{\partial t} - u \frac{\partial \rho}{\partial t}$$

The vector identity for the divergence of the product of a scalar times a vector, we have.

$$\nabla \cdot (\rho u \vec{V}) = u \nabla \cdot (\rho \vec{V}) + (\rho \vec{V} \cdot \nabla u)$$

$$\rho \vec{V} \cdot \nabla u = \nabla \cdot (\rho u \vec{V}) - u \nabla \cdot (\rho \vec{V})$$

So,

$$\rho \frac{Du}{Dt} = \frac{\partial (\rho u)}{\partial t} - u \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u \vec{V}) - u \nabla \cdot (\rho \vec{V})$$

$$\rho \frac{Du}{Dt} = \frac{\partial (\rho u)}{\partial t} - u \left[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) \right] + \nabla \cdot (\rho u \vec{V})$$

Substituting continuity eqn results.

$$\rho \frac{Du}{Dt} = \frac{\partial(\rho u)}{\partial t} + \nabla \cdot (\rho u \vec{V})$$

Substitute this in non-conservation form, we obtain

$$\frac{\partial(\rho u)}{\partial t} + \nabla \cdot (\rho u \vec{V}) = \rho f_x - \frac{\partial P}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}$$

III ly.

$$\frac{\partial(\rho v)}{\partial t} + \nabla \cdot (\rho v \vec{V}) = \rho f_y - \frac{\partial P}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}.$$

$$\frac{\partial(\rho w)}{\partial t} + \nabla \cdot (\rho w \vec{V}) = \rho f_z - \frac{\partial P}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial} + \frac{\partial \tau_{zz}}{\partial z}.$$

These equations are the Navier - Stokes equations in conservation form.

## Energy Equation.

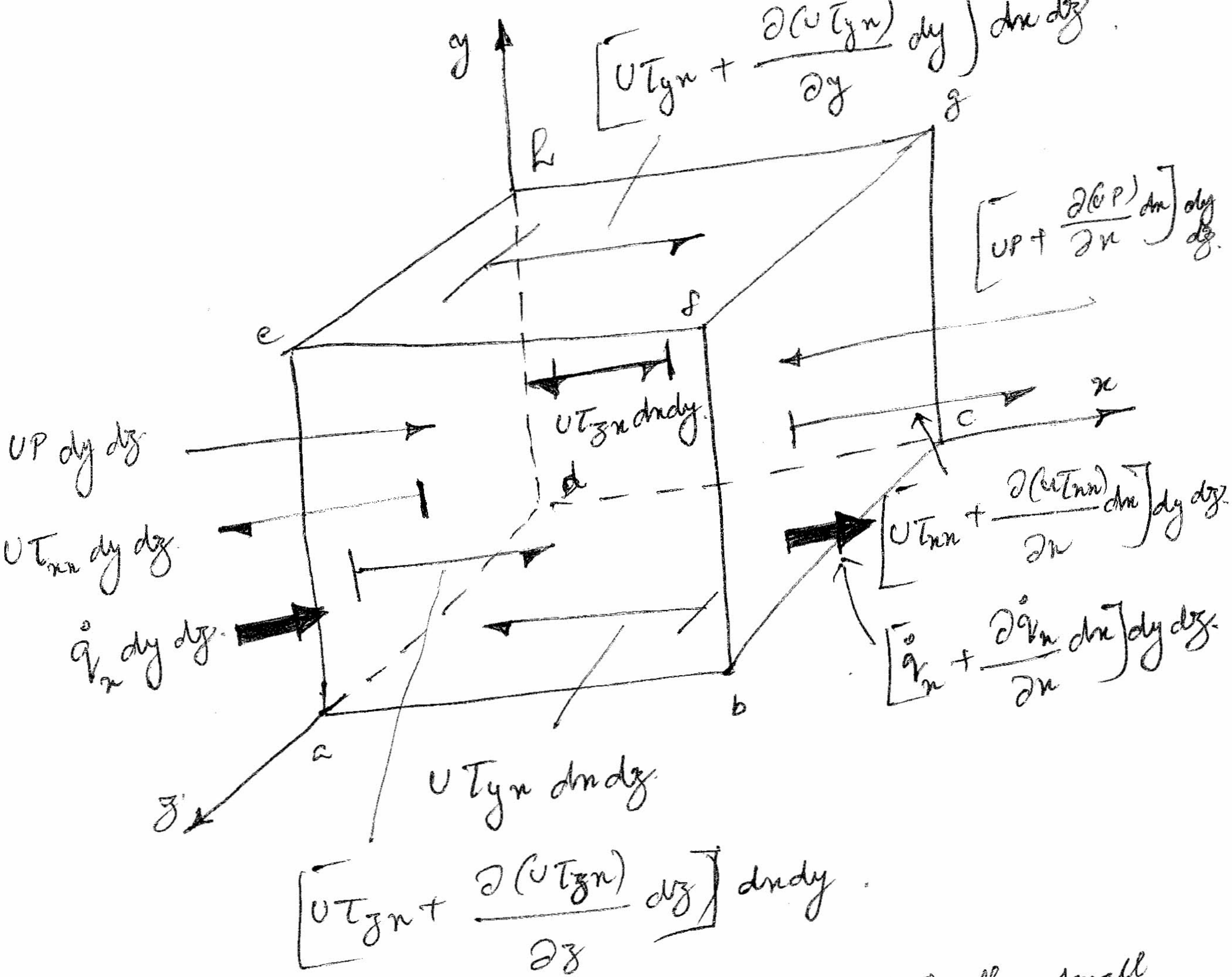
"The Energy is conserved", is the physical principle of the energy equation. Let us consider the first law of thermodynamics.

$$\left\{\begin{matrix}\text{Rate of change of}\\ \text{energy inside the}\\ \text{fluid element}\end{matrix}\right\} = \left\{\begin{matrix}\text{Net flux of}\\ \text{heat into}\\ \text{the element}\end{matrix}\right\} + \left\{\begin{matrix}\text{Rate of work done on}\\ \text{the element due to}\\ \text{body and surface forces}\end{matrix}\right\}$$

$$\therefore \quad A = B + C.$$

The rate of doing work by a force exerted on a moving body is equal to the product of the force and the component of velocity in the direction of force. Hence the rate of work done by the body force acting on the fluid element moving at a velocity $\vec{V}$ is

$$\rho \vec{f} \cdot \vec{V} \,(dx\,dy\,dz).$$




Energy fluxes associated with an infinitesimally small, moving fluid element.

$\dot{q} \rightarrow$ The rate of volumetric heat addition per unit mass.

work done by the surface force on the moving fluid element in $x$-direction,

$$= \left[ \cancel{up} - \cancel{up} - \frac{\partial (up)}{\partial x} dx \right] dy\, dz$$

$$+ \left[ \cancel{u\tau_{xx}} + \frac{\partial (u\tau_{xx})}{\partial x} dx - \cancel{u\tau_{xx}} \right] dy\, dz$$

$$+ \left[ \cancel{u\tau_{yx}} + \frac{\partial (u\tau_{yx})}{\partial y} dy - \cancel{u\tau_{yx}} \right] dx\, dz$$

$$+ \left[ \cancel{u\tau_{zx}} + \frac{\partial (u\tau_{zx})}{\partial z} dz - \cancel{u\tau_{zx}} \right] dx\, dy$$

$$= \left[ -\frac{\partial (up)}{\partial x} + \frac{\partial (u\tau_{xx})}{\partial x} + \frac{\partial (u\tau_{yx})}{\partial y} + \frac{\partial (u\tau_{zx})}{\partial z} \right] dx\,dy\,dz.$$

With the addition in $y$ & $z$ direction, & body force, the net rate of work done is,

$$C = \left[ -\left( \frac{\partial (up)}{\partial x} + \frac{\partial (vp)}{\partial y} + \frac{\partial (wp)}{\partial z} \right) + \rho \vec{f} \cdot \vec{V} \right.$$

$$+ \frac{\partial (u\tau_{xx})}{\partial x} + \frac{\partial (u\tau_{yx})}{\partial y} + \frac{\partial (u\tau_{zx})}{\partial z}$$

$$+ \frac{\partial (v\tau_{xy})}{\partial x} + \frac{\partial (v\tau_{yy})}{\partial y} + \frac{\partial (v\tau_{zy})}{\partial z}$$

$$\left. + \frac{\partial (w\tau_{xz})}{\partial x} + \frac{\partial (w\tau_{yz})}{\partial y} + \frac{\partial (w\tau_{zz})}{\partial z} \right] dx\,dy\,dz.$$

The heat flux is due to.

i) Volumetric heating such as absorbtion or emission of radiation.

ii) Heat transfer across the surface due to temperature gradients i.e. Thermal condution.

$$\left\{ \begin{array}{l} \text{Volumetric heating} \\ \text{of the element} \end{array} \right\} = \rho \dot{q} \, dx\,dy\,dz.$$

The net heat transferred in the x-direction into the fluid element by thermal conduction is,

$$= \left[ \dot{q}_x - \dot{q}_x - \frac{\partial \dot{q}_x}{\partial x} dx \right] dy\,dz.$$

$$= - \frac{\partial \dot{q}_x}{\partial x} dx\,dy\,dz.$$

Taking into account heat transfer in the y & z directions,

$$\left\{ \begin{array}{l} \text{Heating of the} \\ \text{fluid element by} \\ \text{thermal conduction} \end{array} \right\} = - \left( \frac{\partial \dot{q}_x}{\partial x} + \frac{\partial \dot{q}_y}{\partial y} + \frac{\partial \dot{q}_z}{\partial z} \right) dx\,dy\,dz.$$

Then.

$$B = \left[ \rho \dot{q} - \left( \frac{\partial \dot{q}_x}{\partial x} + \frac{\partial \dot{q}_y}{\partial y} + \frac{\partial \dot{q}_z}{\partial z} \right) \right] dx\,dy\,dz.$$

Heat transfer by thermal conduction is proportional to the local temperature gradient.

$$\dot{q}_x = -k\frac{\partial T}{\partial x} \qquad \dot{q}_y = -k\frac{\partial T}{\partial y} \qquad \dot{q}_z = -k\frac{\partial T}{\partial z}.$$

k - Thermal conductivity.

Then,

$$B = \left[\rho\dot{q} + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)\right] dx\,dy\,dz.$$

The time rate of change of energy of the fluid element is the total energy of a moving fluid per unit mass. The total energy is the sum of its internal energy per unit mass, $e$, and its kinetic energy per unit mass, $\frac{V^2}{2}$.

$$\therefore \quad A = \rho\frac{D}{Dt}\left(e + \frac{V^2}{2}\right) dx\,dy\,dz$$

The final form of the energy equation.

$$\rho\frac{D}{Dt}\left(e + \frac{V^2}{2}\right) = \rho(\dot{q} + \vec{f}\cdot\vec{V}) + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)$$
$$- \left(\frac{\partial(up)}{\partial x} + \frac{\partial(vp)}{\partial y} + \frac{\partial(wp)}{\partial z}\right) + \frac{\partial(u\tau_{xx})}{\partial x} + \frac{\partial(u\tau_{yx})}{\partial y}$$
$$+ \frac{\partial(u\tau_{zx})}{\partial z} + \frac{\partial(v\tau_{xy})}{\partial x} + \frac{\partial(v\tau_{yy})}{\partial y} + \frac{\partial(v\tau_{zy})}{\partial z}$$
$$+ \frac{\partial(w\tau_{xz})}{\partial x} + \frac{\partial(w\tau_{yz})}{\partial y} + \frac{\partial(w\tau_{zz})}{\partial z}.$$

This is the non-conservation form of the energy equation. It is derived in the form of total energy. Frequently, the energy equation is written in a form that involves just the internal energy.

Let us consider the non-conservational form of Navier stokes equations and multiply by $u$, $v$ and $w$ in $x$, $y$ and $z$ direction respectively.

$$\rho \frac{D\left(\frac{u^2}{2}\right)}{Dt} = \rho u f_x - u\frac{\partial p}{\partial x} + u\frac{\partial \tau_{xx}}{\partial x} + u\frac{\partial \tau_{yx}}{\partial y} + u\frac{\partial \tau_{zx}}{\partial z}.$$

$$\rho \frac{D\left(\frac{v^2}{2}\right)}{Dt} = \rho v f_y - v\frac{\partial p}{\partial y} + v\frac{\partial \tau_{xy}}{\partial x} + v\frac{\partial \tau_{yy}}{\partial y} + v\frac{\partial \tau_{zy}}{\partial z}.$$

$$\rho \frac{D\left(\frac{w^2}{2}\right)}{Dt} = \rho w f_z - w\frac{\partial p}{\partial z} + w\frac{\partial \tau_{xz}}{\partial x} + w\frac{\partial \tau_{yz}}{\partial y} + w\frac{\partial \tau_{zz}}{\partial z}.$$

Adding above eqn gives. (where, $u^2+v^2+w^2 = V^2$).

$$\rho \frac{D\left(\frac{V^2}{2}\right)}{Dt} = \rho\left(uf_x + vf_y + wf_z\right) - \left(u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z}\right)$$

$$+ u\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)$$

$$+ v\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) + w\left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right)$$

Subtract above with the non-conservation form.

$$
\begin{aligned}
\rho \frac{D}{Dt}\left(e+\cancel{\frac{V^2}{2}}-\cancel{\frac{V^2}{2}}\right) = {} & \rho\left(\dot{q}+\cancel{\vec{f}\cdot\vec{V}}-\cancel{\vec{f}\cdot\vec{V}}\right) + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) \\
& + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) - \cancel{u\frac{\partial p}{\partial x}} - p\frac{\partial u}{\partial x} - \cancel{v\frac{\partial p}{\partial y}} \\
& - p\frac{\partial v}{\partial y} - \cancel{w\frac{\partial p}{\partial z}} - p\frac{\partial w}{\partial z} + \cancel{u\frac{\partial \tau_{xx}}{\partial x}} \\
& + \tau_{xx}\frac{\partial u}{\partial x} + \cancel{u\frac{\partial \tau_{yx}}{\partial y}} + \tau_{yx}\frac{\partial u}{\partial y} \\
& + \cancel{u\frac{\partial \tau_{zx}}{\partial z}} + \tau_{zx}\frac{\partial u}{\partial z} + \cancel{v\frac{\partial \tau_{xy}}{\partial x}} \\
& + \tau_{xy}\frac{\partial v}{\partial x} + \cancel{v\frac{\partial \tau_{yy}}{\partial y}} + \tau_{yy}\frac{\partial v}{\partial y} \\
& + \cancel{v\frac{\partial \tau_{zy}}{\partial z}} + \tau_{zy}\frac{\partial v}{\partial z} + \cancel{w\frac{\partial \tau_{xz}}{\partial x}} \\
& + \tau_{xz}\frac{\partial w}{\partial x} + \cancel{w\frac{\partial \tau_{yz}}{\partial y}} + \tau_{yz}\frac{\partial w}{\partial y} \\
& + \cancel{w\frac{\partial \tau_{zz}}{\partial z}} + \tau_{zz}\frac{\partial w}{\partial z} + \cancel{u\frac{\partial p}{\partial x}} \\
& + \cancel{v\frac{\partial p}{\partial y}} + \cancel{w\frac{\partial p}{\partial z}} - \cancel{u\left(\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}\right)} \\
& - \cancel{v\left(\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}\right)} \\
& - \cancel{w\left(\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \tau_{zz}}{\partial z}\right)}.
\end{aligned}
$$

$$\rho \frac{De}{Dt} = \rho \dot{q} + \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right)$$

$$- p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \tau_{xx}\frac{\partial u}{\partial x} + \tau_{yx}\frac{\partial u}{\partial y}$$

$$+ \tau_{zx}\frac{\partial u}{\partial z} + \tau_{xy}\frac{\partial v}{\partial x} + \tau_{yy}\frac{\partial v}{\partial y} + \tau_{zy}\frac{\partial v}{\partial z}$$

$$+ \tau_{xz}\frac{\partial w}{\partial x} + \tau_{yz}\frac{\partial w}{\partial y} + \tau_{zz}\frac{\partial w}{\partial z}.$$

Note that the body force terms have cancelled. The energy equation when written in terms of $e$ does not explicitly contain the body force.

W.K.T $\quad \tau_{xy} = \tau_{yx}$, $\tau_{xz} = \tau_{zx}$, $\tau_{yz} = \tau_{zy}$.

Then.

$$\rho \frac{De}{Dt} = \rho \dot{q} + \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right)$$

$$- p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \tau_{xx}\frac{\partial u}{\partial x} + \tau_{yy}\frac{\partial v}{\partial y}$$

$$+ \tau_{zz}\frac{\partial w}{\partial z} + \tau_{xy}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$+ \tau_{xz}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) + \tau_{yz}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right).$$

The energy equation in conservation form can be obtained as follows.

From the definition of the substantial derivative

$$\rho \frac{De}{Dt} = \rho \frac{\partial e}{\partial t} + \rho \vec{V}.\nabla e$$

W.K.T

$$\frac{\partial(\rho e)}{\partial t} = \rho \frac{\partial e}{\partial t} + e \frac{\partial \rho}{\partial t}$$

Then

$$\rho \frac{\partial e}{\partial t} = \frac{\partial(\rho e)}{\partial t} - e \frac{\partial \rho}{\partial t}$$

From the vector identity concerning the divergence of the product of a scalar times a vector.

$$\nabla.(\rho e \vec{V}) = e \nabla.(\rho \vec{V}) + \rho \vec{V}.\nabla e.$$

Then

$$\rho \vec{V}.\nabla e = \nabla.(\rho e \vec{V}) - e \nabla.(\rho \vec{V})$$

Then the substantial derivative of this equation

$$\rho \frac{De}{Dt} = \frac{\partial(\rho e)}{\partial t} - e\left[\frac{\partial \rho}{\partial t} + \nabla.(\rho \vec{V})\right] + \nabla.(\rho e \vec{V})$$

from continuity eqns.

$$\frac{\rho De}{Dt} = \frac{\partial(\rho e)}{\partial t} + \nabla.(\rho e \vec{V}),$$

Then, the conservation form of the energy equation is

$$\frac{\partial(\rho e)}{\partial t} + \nabla\cdot(\rho e\vec{V}) = \rho\dot{q} + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)$$
$$- p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \tau_{xx}\frac{\partial u}{\partial x} + \tau_{yy}\frac{\partial v}{\partial y}$$
$$+ \tau_{zz}\frac{\partial w}{\partial z} + \tau_{xy}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$
$$+ \tau_{yz}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) + \tau_{xz}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right).$$

<u>Basic aspects of Incompressible, Inviscid flow.</u>

Incompressible flow is constant density flow, i.e $\rho = c$.

Visualize a fluid element of fixed mass moving along a streamline in an incompressible flow. Its density is constant, then the volume of the fluid element is also constant.

W.K.T,

$\nabla\cdot\vec{V}$ is the time rate of change of the volume of a fluid element per unit volume. Since the volume is constant for a fluid element in incompressible flow,

$$\nabla\cdot\vec{V} = 0$$

Further, if the fluid element does not rotate as it moves along the streamline, then the flow is called irrotational (translational). For such flow, the velocity can be expressed as the gradient of a scalar function called the velocity potential, $\phi$.

$$\vec{V} = \nabla \phi$$

Then,

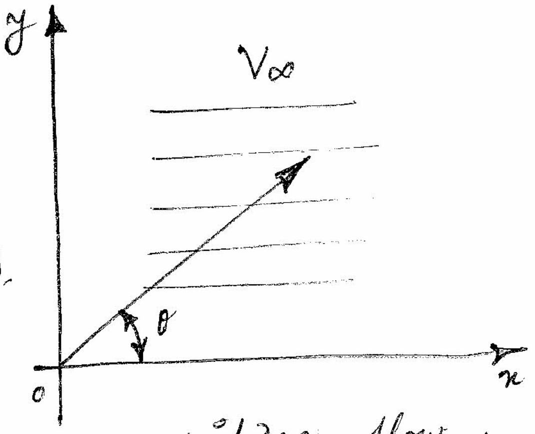
$$\nabla . \nabla \phi = 0$$

$$\boxed{\nabla^2 \phi = 0 .}$$

This is Laplace's equation and is linear. Hence any number of particular solutions can be added together to obtain another solution. This establishes a basic philosophy of the solution of incompressible flows, namely, that a complicated flow pattern for an irrotational, incompressible flow can be synthesized by adding together a number of elementary flows which are also irrotational and incompressible.

## Uniform flow.

Consider a uniform flow with velocity $V_\infty$ moving in the x-direction.



Uniform flow.

This flow is irrotational and a solution of Laplace's equation for uniform flow yields,

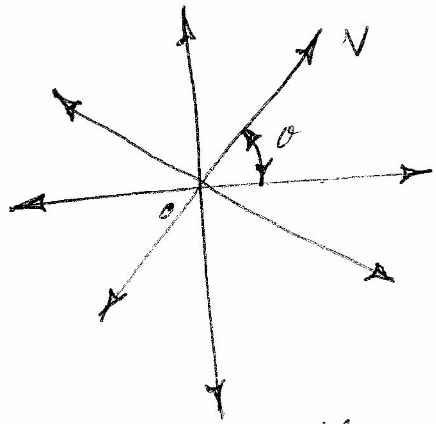
$$\boxed{\phi = V_\infty x.}$$

In polar coordinates,

$$\boxed{\phi = V_\infty r \cos\theta.}$$

## Source flow

Consider a flow with straight streamlines emanating from a point, where the velocity along each streamline varies inversely with distance from the point. Such flow is called source flow.



Source flow

This flow is also irrotational and a solution of Laplace's equation yields

$$\boxed{\phi = \frac{\Lambda}{2\pi} \ln r}$$

where $\Lambda$ is defined as the source strength.

$\Lambda$ is physically the rate of volume flow from the source, per unit depth perpendicular to the paper. The line perpendicular to the paper is a line source with strength $\Lambda$ per unit length.

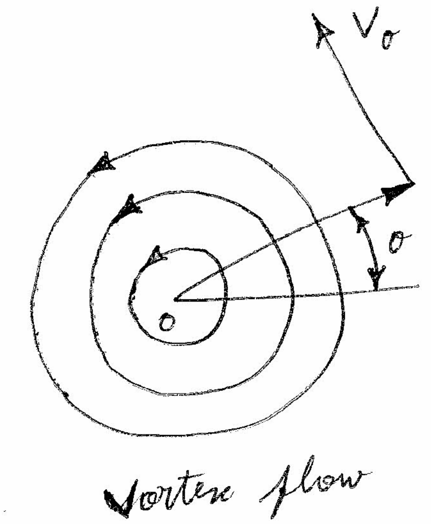
$$\Lambda = \frac{m^{\cancel{3}2}}{sec.\cancel{m}} = \frac{m^2}{sec.}$$

## Vortex flow.

consider a flow where all the streamlines are concentric circles about a given point, where the velocity along each streamline is inversely proportional to the distance from the centre. Such flow is called vortex flow. This flow is irrotational, and a solution of Laplace's equation yields.

$$\boxed{\phi = -\frac{\Gamma}{2\pi}\theta}$$

where, $\Gamma$ is the strength of the vortex. The point $o$ is simply one point formed by the intersection of the plane of the paper and a line perpendicular to the paper.



Vortex flow

This line is called a vortex filament of strength $\Gamma$. The strength $\Gamma$ is the circulation around the vortex filament, where circulation is defined as

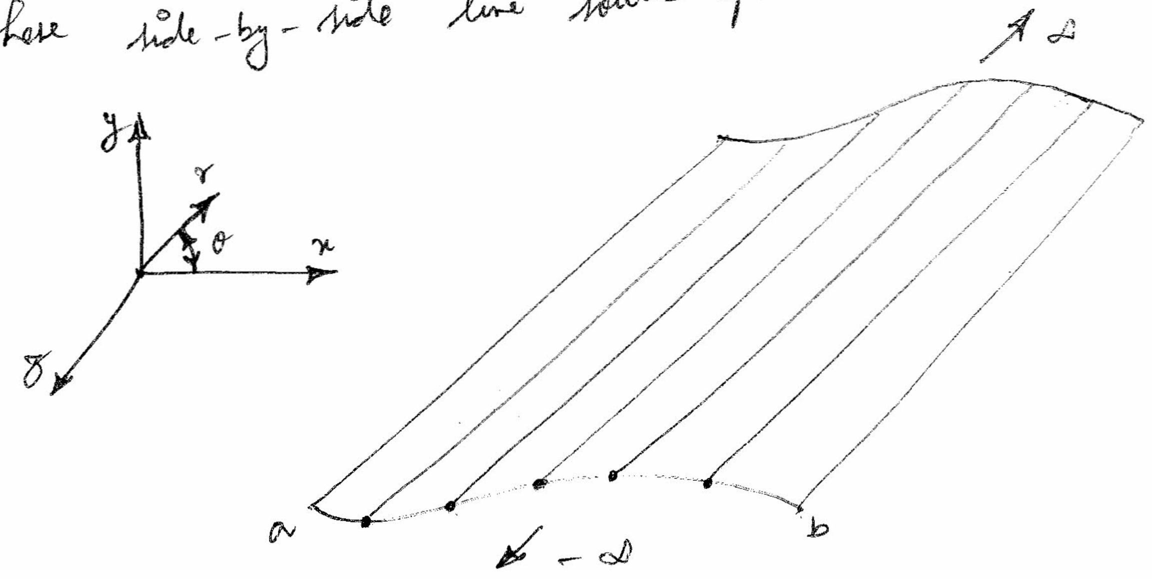
$$\Gamma = -\oint \vec{V} \cdot d\vec{s}$$

This is the general definition of circulation. For a vortex filament, the above expression for $\Gamma$ is defined as the vortex strength.

## The Source Panel Method:

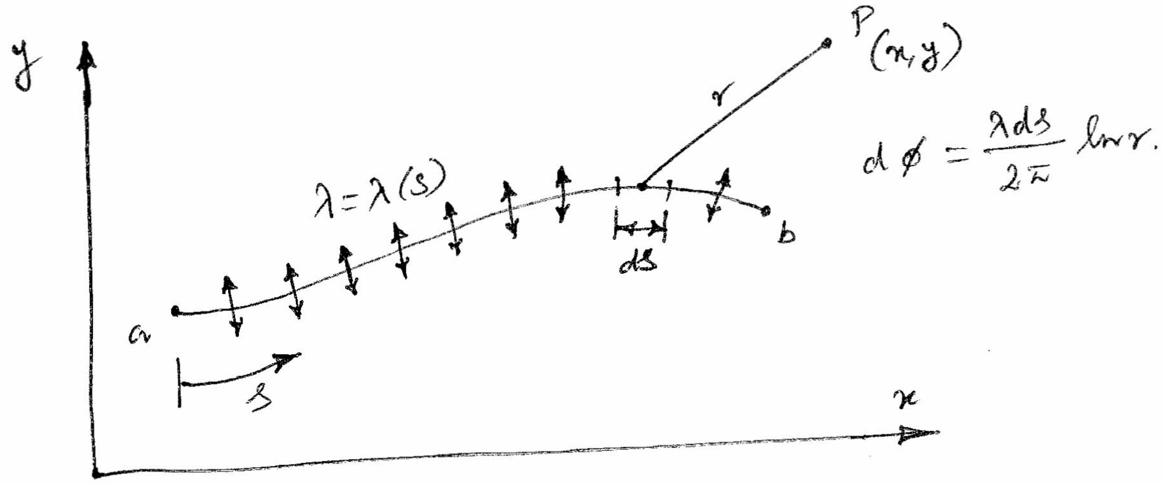
(Non-lifting flows over arbitrary two-dimensional bodies)

Consider an ~~single~~ infinite number of line sources side by side, where the strength of each line source is infinitesimally small. These side-by-side line sources form a source sheet.




Source sheet.

If we look along the series of line sources, the source sheet will appear as.




Edge view

$s \rightarrow$ distance measured along the source sheet

$\lambda = \lambda(s) \rightarrow$ Source strength per unit length $\left( \frac{m^{\cancel{2}}}{s\, \cancel{m}} = \frac{m}{s} \right)$

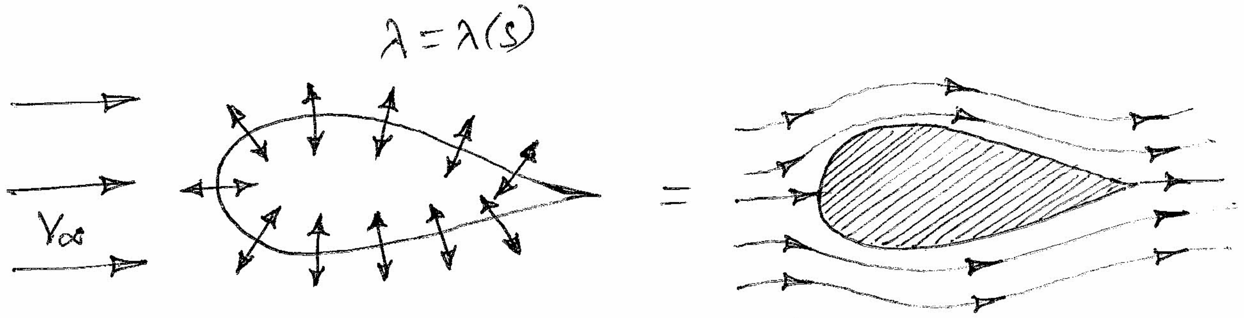
Therefore, the strength of an infinitesimal portion ds of the sheet is $\lambda\, ds$. Now consider point P in the flow, located at distance r from ds. The small section of the source sheet of strength $\lambda\, ds$ induces an infinitesimally small potential, $d\phi$ at point P and is given by

$$d\phi = \frac{\lambda\, ds}{2\pi} \ln r. \longrightarrow (1)$$

The complete velocity potential at point P, induced by the entire source sheet from a to b

$$\phi(x,y) = \int_a^b \frac{\lambda\, ds}{2\pi} \ln r \longrightarrow (2).$$

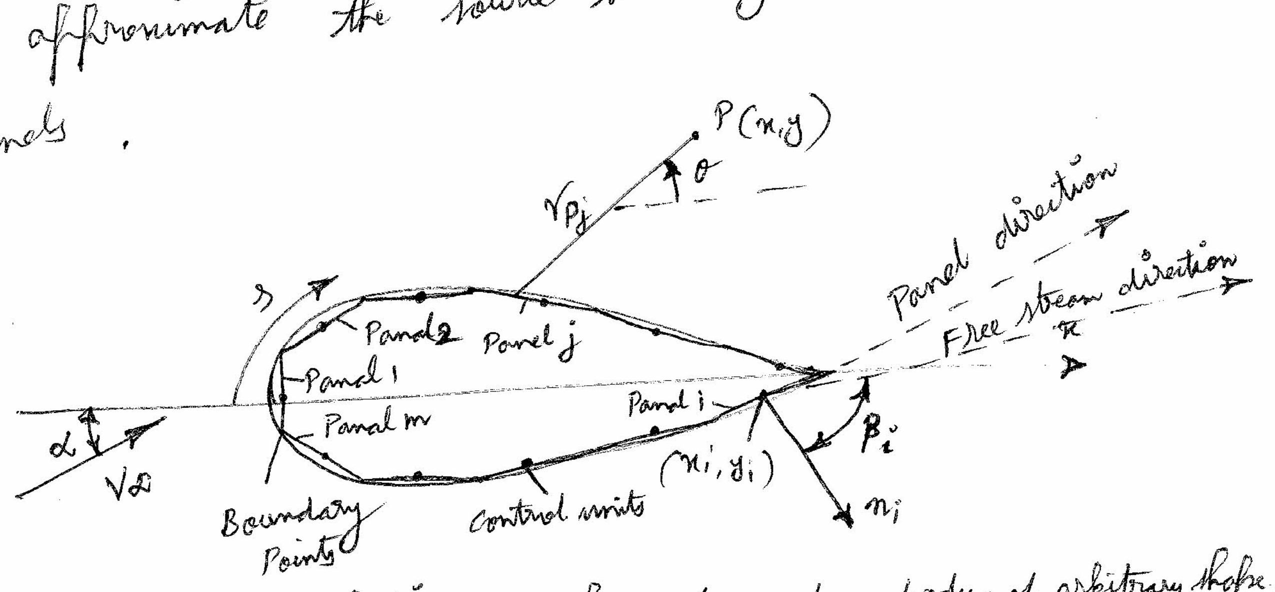
Now consider a given body of arbitrary shape in a flow with free-stream velocity $V_\infty$.




Superposition of a uniform flow and a source sheet on a body of given shape, to produce flow over body.

Let us cover the surface of the prescribed body with a source sheet, where the strength $\lambda(s)$ varies in such a fashion that the combined action of the uniform flow and the source sheet makes the airfoil surface a streamline of the flow.

In order to find the appropriate $\lambda(s)$, let us approximate the source sheet by a series of straight panels.




Source panel distribution over the surface of a body of arbitrary shape.

The source strength $\lambda$ is varying from one panel to the next. That is, if there is a total of $n$ panels the source panel strengths are $\lambda_1, \lambda_2, \dots \lambda_j, \dots \lambda_n$. These Panel strengths are unknown and this panel technique is to solve for $\lambda$'s such that the body surface becomes a streamline of the flow. This boundary condition is imposed numerically by defining the midpoint of each panel to be a control point.

Let P be a point located at $(x,y)$ in the flow, and let $r_{pj}$ be the distance from any point on the $j^{th}$ Panel to P. The velocity potential induced at P due to the $j^{th}$ panel

$$\Delta\phi_j = \frac{\lambda_j}{2\pi}\int_j \ln r_{pj}\, ds_j \rightarrow (3)$$

The potential at P due to all the panels is the summation over all the panels.

$$\phi(P) = \sum_{j=1}^{n} \Delta\phi_j = \sum_{j=1}^{n} \frac{\lambda_j}{2\pi}\int_j \ln r_{pj}\, ds_j \rightarrow (4)$$

The distance $r_{pj}$ is given by

$$r_{pj} = \sqrt{(x-x_j)^2 + (y-y_j)^2}$$

Since point P is just an arbitrary point in the flow, let us put P at the control point of the $i^{th}$ panel.

Then,

$$\phi(x_i, y_i) = \sum_{j=1}^{n} \frac{\lambda_j}{2\pi} \int_j \ln r_{ij} \, ds_j \rightarrow (5)$$

and

$$r_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$$

This equation is physically the contribution of all the panels to the potential at the control point of the $i^{th}$ panel.

From boundary condition, the normal component of the flow velocity is zero at the control points. The free-stream velocity at some incidence angle $\alpha$ to the x-axis is

$$V_{\infty,n} = \vec{V}_\infty \cdot \vec{n}_i = V_\infty \cos\beta_i$$

The normal component of velocity induced at $(x_i, y_i)$ by the source panels is

$$V_n = \frac{\partial}{\partial n_i}\left[\phi(x_i, y_i)\right]$$

when $j = i$, the contribution to the derivative is simply $\lambda_i/2$.

$$\therefore \quad V_n = \frac{\lambda_i}{2} + \sum_{j=1}^{n} \frac{\lambda_j}{2\pi} \int_j \frac{\partial}{\partial n_i} (\ln r_{ij})\, ds_j \rightarrow ⑥$$

(velocity)

The boundary conditions states that this sum must be zero.

$$V_{\infty,n} + V_n = 0$$

$$\therefore \quad \frac{\lambda_i}{2} + \sum_{j=1}^{n} \frac{\lambda_j}{2\pi} \int_j \frac{\partial}{\partial n_i} (\ln r_{ij})\, ds_j + V_\infty \cos\beta_i = 0 \rightarrow ⑦$$

(crux)

The above eqn is a point of difficulty of the source panel method. The values of the integrals are depend simply on the panel geometry. Let $I_{ij}$ be the value of this integral on the $i^{th}$ panel.

$$\frac{\lambda_i}{2} + \sum_{j=1}^{n} \frac{\lambda_j}{2\pi} I_{ij} + V_\infty \cos\beta_i = 0 \rightarrow ⑧$$

This is a linear algebraic equation with $n$ unknowns, $\lambda_1, \lambda_2, \ldots \lambda_n$. Now apply the boundary condition to the control point of all the panels, let $i = 1, 2, \ldots n$.

Then the velocity tangent to the surface at each control point can be calculated as

$$V_{\infty,s} = V_\infty \sin\beta_i$$

$s \rightarrow$ distance along the body surface.

The tangential velocity $V_s$ at the control point of the $i^{th}$ panel induced by all the panels is obtained by differentiating eqn (5) w.r.t $s$.

$$V_s = \frac{\partial \phi}{\partial s} = \sum_{j=1}^{n} \frac{\lambda_j}{2\pi} \int_j \frac{\partial}{\partial s} (\ln r_{ij})\, ds_j \rightarrow (9)$$

The total surface velocity at the $i^{th}$ panel control point $V_i$ is the sum of the contribution from the freestream and from the source panels.

$$\therefore V_i = V_{\infty,s} + V_s = \sum_{j=1}^{n} \frac{\lambda_j}{2\pi} \int_j \frac{\partial}{\partial s} (\ln r_{ij})\, ds_j + V_\infty \sin\beta_i \rightarrow (10)$$

In turn, the pressure co-efficient at the $i^{th}$ control point is obtained from Bernoulli's equation.

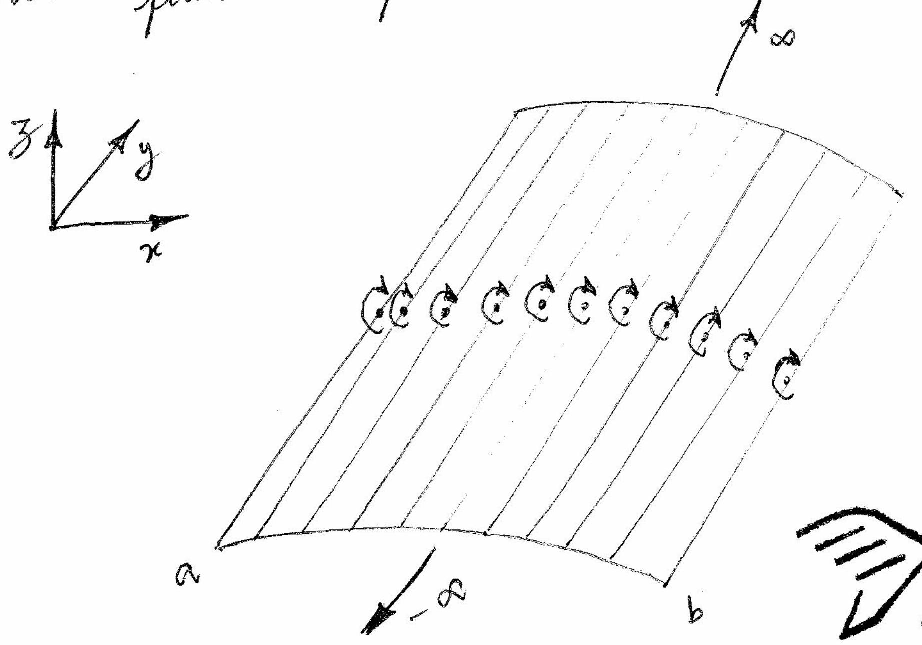
$$C_{p,i} = 1 - \left(\frac{V_i}{V_\infty}\right)^2 \longrightarrow (11)$$

In this fashion, the source panel method gives the pressure distribution over the surface of a non-lifting body of arbitrary shape.

## The Vortex Panel Method

(Lifting flows over arbitrary Two-dimensional bodies.

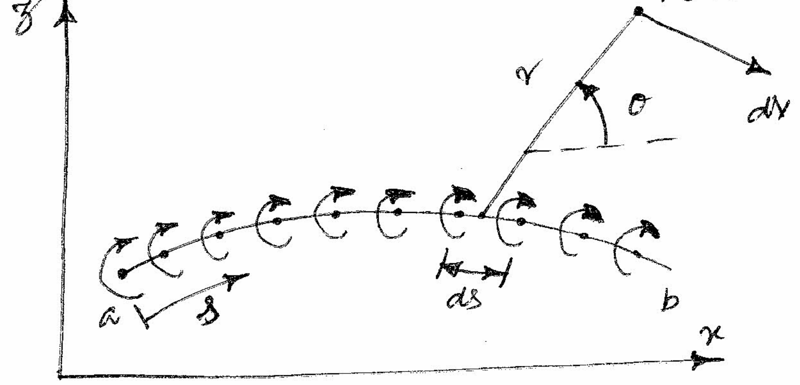
Imagine an infinite number of straight vortex filaments side-by-side, where the strength of each filament is infinitesimally small. These side-by-side vortex filaments form a vortex sheet.






Vortex sheet.

If we look along the series of vortex filaments,




Edge View

Let $s$ be the distance measured along the vortex sheet in the edge view. $\gamma = \gamma(s)$ is the strength of the vortex sheet, per unit length along $s$. Thus, the strength of an infinitesimal portion $ds$ of the sheet is $\gamma ds$.

Now consider point P in the flow, located a distance $r$ from $ds$. The small section of the vortex sheet of strength $\gamma ds$ induces a velocity potential at P, obtained as

$$d\phi = -\frac{\gamma ds}{2\pi}\theta \longrightarrow (1)$$

The velocity potential at P due to the entire vortex sheet from a to b is

$$\phi = -\frac{1}{2\pi}\int_a^b \theta\gamma ds \longrightarrow (2)$$

In addition, the circulation around the vortex sheet is the sum of the strengths of the elemental vortices.

$$\Gamma = \int_a^b \gamma ds. \longrightarrow (3)$$

Another property of a vortex sheet is that the component of flow velocity tangential to the sheet experiences a discontinuous change across the sheet, given by

$$\gamma = u_1 - u_2 \rightarrow (4)$$

where, $u_1$ and $u_2$ are the tangential velocities just above and below the sheet respectively. Eqn (4) is used to demonstrate that, for flow over an airfoil, the value of $\gamma$ is zero at the trailing edge of the airfoil.

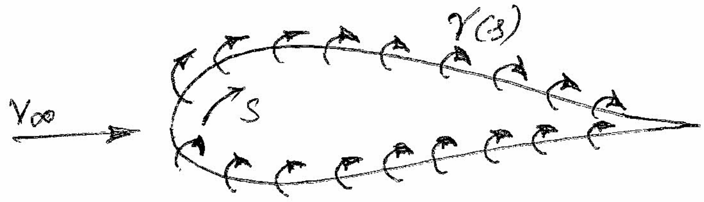
$$\gamma_{TE} = 0 \rightarrow (5)$$

This is one form of the Kutta condition which fixes the precise value of the circulation around an airfoil with a sharp trailing edge.

W.K.T the Kutta-Joukowski theorem states that the circulation around the sheet is related to the lift force on the sheet

$$L = \rho_\infty V_\infty \Gamma \rightarrow (6)$$

consider an arbitrary two-dimensional body. Let us wrap a vortex sheet over the complete surface of the body and approximate the vortex sheet by a series of straight panels.




Simulation of an arbitrary airfoil by distributing a vortex sheet over the airfoil surface.

Let P be a point located at $(x, z)$ in the flow, and let $r_{pj}$ be the distance from any point on the $j^{th}$ panel to P. The radius $r_{pj}$ makes the angle $\theta_{pj}$ with respect to $x$-axis. The velocity potential induced at P due to the $j^{th}$ panel

$$\Delta \phi_j = -\frac{1}{2\pi} \int_j \theta_{pj} \gamma_j \, ds_j \longrightarrow (7)$$

The angle $\theta_{pj}$ is given by

$$\theta_{pj} = \tan^{-1}\left(\frac{z - z_j}{x - x_j}\right) \longrightarrow (8)$$

$\gamma_j$ is constant over the $j^{th}$ panel and the integral is taken over the $j^{th}$ panel only. The potential at P due to all the panels is summation over all the panels.

$$\phi(P) = \sum_{j=1}^{n} \phi_j = -\sum_{j=1}^{n} \frac{\gamma_j}{2\pi} \int_j \theta_{pj} \, ds_j \longrightarrow (9)$$

Since Point P is just an arbitrary point in the flow, let us put P at the control point of the i th panel. The coordinates of this control point are $(x_i, z_i)$

$$\theta_{i,j} = \tan^{-1}\left(\frac{z_i - z_j}{x_i - x_j}\right) \longrightarrow (10)$$

and

$$\phi(x_i, z_i) = -\sum_{j=1}^{n} \frac{\gamma_j}{2\pi} \int_j \theta_{ij}\, ds_j \longrightarrow (11).$$

Eqn (11) is physically the contribution of all the panels to the potential at the control point of the i th panel. At the control points, the normal component of the velocity is zero. This velocity is the superposition of the uniform flow velocity and the velocity induced by all the vortex panels. The component of $V_\infty$ normal to the i th panel is given by

$$V_{\infty,n} = V_\infty \cos\beta_i$$

The normal component of velocity induced at $(x_i, z_i)$ by the vortex panels is

$$V_n = \frac{\partial}{\partial n_i}\left[\phi(x_i, z_i)\right]$$

~~Then~~ Substituting eq. 11 brings with the summation

$$V_n = -\sum_{j=1}^{n} \frac{\gamma_j}{2\pi} \int_j \frac{\partial \theta_{ij}}{\partial n_i}\, ds_j \rightarrow (12)$$

The boundary condition states that velocities sum must be zero.

$$V_{\infty,n} + V_n = 0$$

Then,

$$V_\infty \cos\beta_i - \sum_{j=1}^{n} \frac{\gamma_j}{2\pi} \int_j \frac{\partial \theta_{ij}}{\partial n_i}\, ds_j = 0. \rightarrow (13)$$

The above eqn is the point of difficulty of the vortex panel method. The values of the integrals are depend simply on the panel geometry. Let $J_{i,j}$ be the value of this integral on the $i^{th}$ panel.

$$V_\infty \cos\beta_i - \sum_{j=1}^{n} \frac{\gamma_j}{2\pi} J_{i,j} = 0 \rightarrow (14)$$

Eqn (14) is a linear algebraic equation with $n$ unknowns, $\gamma_1, \gamma_2, .. \gamma_n$. For the ~~lifting~~ lifting case with vortex panels, the Kutta condition must be satisfied. The total circulation and the resulting lift are obtained as follows.

Let $S_j$ be the length of the $j^{th}$ panel. Then the circulation due to the $j^{th}$ panel is $\gamma_j S_j$. So, the total circulation due to all the panels is

$$\Gamma = \sum_{j=1}^{n} \gamma_j S_j$$

Hence the lift per unit span is obtained from

$$L' = \rho_\infty V_\infty \sum_{j=1}^{n} \gamma_j S_j$$

This is the general form of vortex panel method.

## Mathematical properties of the fluid dynamic equations

Let us consider a fairly simple system of quasilinear equations. They will not be the flow equations, but they are similar in some respects. Consider the system of quasilinear equations given below.

$$\left.\begin{aligned} a_1 \frac{\partial u}{\partial x} + b_1 \frac{\partial u}{\partial y} + c_1 \frac{\partial v}{\partial x} + d_1 \frac{\partial v}{\partial y} &= f_1 \\ a_2 \frac{\partial u}{\partial x} + b_2 \frac{\partial u}{\partial y} + c_2 \frac{\partial v}{\partial x} + d_2 \frac{\partial v}{\partial y} &= f_2 \end{aligned}\right\} \rightarrow (1)$$

where, $u$ and $v$ are the dependent variables, functions of $x$ and $y$, and the coefficients $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, f_1$ & $f_2$ can be functions of $x, y, u$ and $v$.

Consider any point in the $xy$-plane. Let us seek the lines through this point along which the derivatives of $u$ and $v$ are indeterminant and across which may be discontinuous. Such lines are called characteristic lines. To find such lines, we assume that $u$ and $v$ are continuous,

$$\left.\begin{aligned} \text{Since } u = u(x,y) \qquad du &= \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \\ v = v(x,y) \qquad dv &= \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy \end{aligned}\right\} \rightarrow (2)$$

These four linear equations can be written in matrix form as.

$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ dx & dy & 0 & 0 \\ 0 & 0 & dx & dy \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ du \\ dv \end{bmatrix} \rightarrow (3)$$

Let $[A]$ denote the coefficient matrix. From Cramer's rule, if $|A| \neq 0$, then unique solutions can be obtained for $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ & $\frac{\partial v}{\partial y}$. On other hand, if $|A| = 0$, then $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ & $\frac{\partial v}{\partial y}$ are indeterminant.

We are seeking the particular directions in the $xy$-plane along which these derivatives of $u$ and $v$ are indeterminant. Therefore, let us set $|A| = 0$,

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ dx & dy & 0 & 0 \\ 0 & 0 & dx & dy \end{vmatrix} = 0$$

Hence

$$(a_1 c_2 - a_2 c_1)(dy)^2 - (a_1 d_2 - a_2 d_1 + b_1 c_1 - b_2 c_1)(dx)(dy) + (b_1 d_2 - b_2 d_1)(dx)^2 = 0 \rightarrow (4)$$

divide eqn (4) by $(dx)^2$

$$(a_1 c_2 - a_2 c_1)\left(\frac{dy}{dx}\right)^2 - (a_1 d_2 - a_2 d_1 + b_1 c_2 - b_2 c_1)\frac{dy}{dx} + (b_1 d_2 - b_2 d_1) = 0 \rightarrow (5)$$

Eqn ⑤ is a quadratic equation in $\frac{dy}{dx}$. For any point in the xy-plane the solution will give the slopes of the line along which the derivatives of $u$ and $v$ are indeterminant. let,

$$a = (a_1 c_2 - a_2 c_1)$$

$$b = -(a_1 d_2 - a_2 d_1 + b_1 c_2 - b_2 c_1)$$

$$c = (b_1 d_2 - b_2 d_1)$$

Then eqn ⑤ can be written as

$$a\left(\frac{dy}{dx}\right)^2 + b\left(\frac{dy}{dx}\right) + c = 0 \longrightarrow (6)$$

Hence from the quadratic formula:

$$\frac{dy}{dx} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \longrightarrow (7)$$

Eqn ⑦ gives the direction of the characteristics lines through a given xy point. These lines have a different nature, depending on the value of the discriminant.

$$\text{Discriminant}, \ D = b^2 - 4ac. \longrightarrow (8)$$

The characteristic lines may be real and distinct, real and equal or imaginary depending on the value of D.

If $D > 0$ : Two real and distinct characteristics exist through each point in the $xy$-plane. Here, the system of equations is called hyperbolic.

If $D = 0$ : One real characteristic exists. Here, the system of equations is called parabolic

If $D < 0$ : The characteristic lines are imaginary. Here, the system of equations is called elliptic.

The classification of quasilinear partial differential equations as either elliptic, parabolic or hyperbolic is common in the analysis of such equations.

The origin of the words elliptic, parabolic or hyperbolic used to label these equation is simply a direct analogy with the case for conic sections. The general equation for a conic section from analytic geometry is

$$ax^2 + bxy + cy^2 + dx + ey + f = 0 \longrightarrow (9)$$

where, if

$b^2 - 4ac > 0$, the conic is a hyperbola

$b^2 - 4ac = 0$, the conic is a parabola

$b^2 - 4ac < 0$, the conic is an ellipse.

We note that, for hyperbolic partial differential equations, the fact that two real and distinct characteristics exist allows the development of a method of the ready solution of these equations.

In eqn (3), if we actually attempt to solve for $\frac{\partial u}{\partial y}$ using Cramer's rule, we have

$$\frac{\partial u}{\partial y} = \frac{|N|}{|A|} = \frac{0}{0}$$

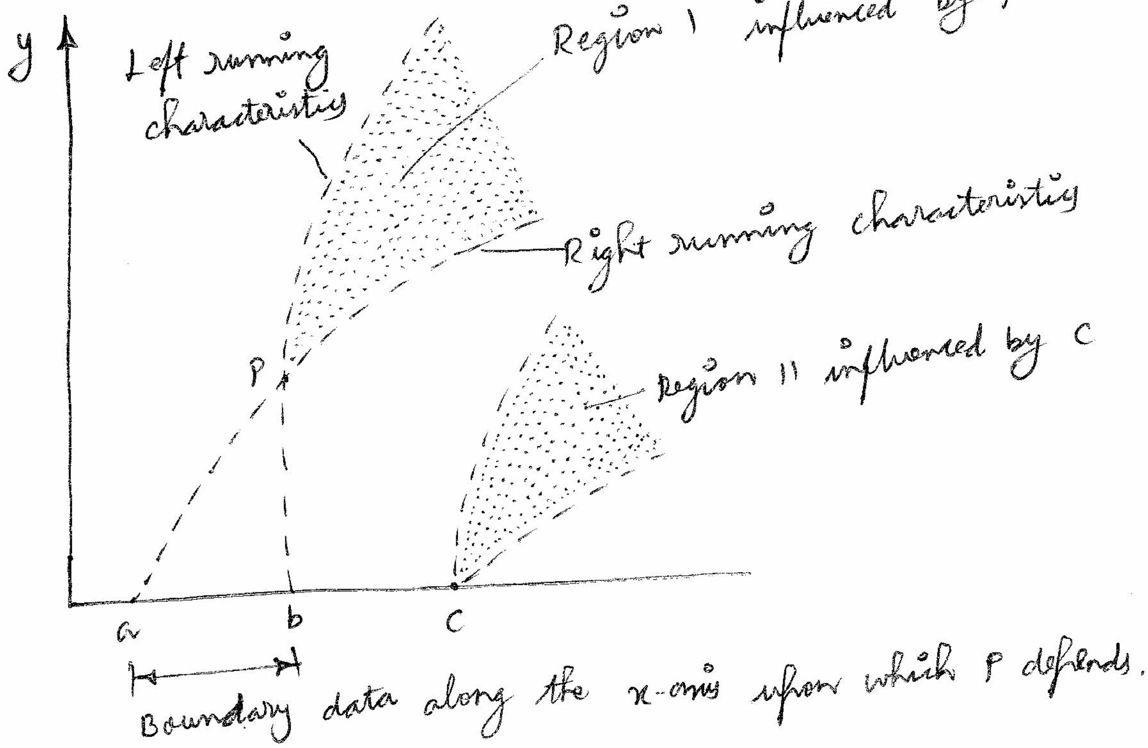
where the numerator determinant is

$$|N| = \begin{vmatrix} a_1 & f_1 & c_1 & d_1 \\ a_2 & f_2 & c_2 & d_2 \\ dx & du & 0 & 0 \\ 0 & dv & dx & dy \end{vmatrix} \longrightarrow (10)$$

The expansion of eqn (10) will lead to equations involving the flowfield variables which are ordinary differential equations and in some cases are algebraic equations and are called the compatibility equations. They hold only along the characteristic lines. This is the essence of solving the original hyperbolic partial differential equation. (Simply integrating ordinary differential equations along the characteristic lines in the xy-plane.) This is called the <u>method of characteristics</u>.

## Hyperbolic Equations

For hyperbolic equations, information at a given point P influences only those regions between the advancing characteristics. For example, examine the following figure which is sketched for a two-dimensional problem with two independent space variables




Domain and boundaries for the solution of hyperbolic equations (Two dimensional steady flow).

Point P is located at a given (x,y). Consider the left and right running characteristics through P. Information at point P influences only the shaded region. (This has a collorary effect on boundary conditions for hyperbolic equation)

Assume that the x-axis is a given boundary condition for the problem, i.e. the dependent variables u and v are known along the x-axis. Then the solution can be obtained by 'marching forward' in the distance y, starting from the given boundary. However the solution for u and v at point P will depend only on that part of the boundary between a and b.

Information at point c, which is outside the internal ab, is propagated along characteristics through c and influences only region II. Point P is outside region II and hence does not feel the information from point c.

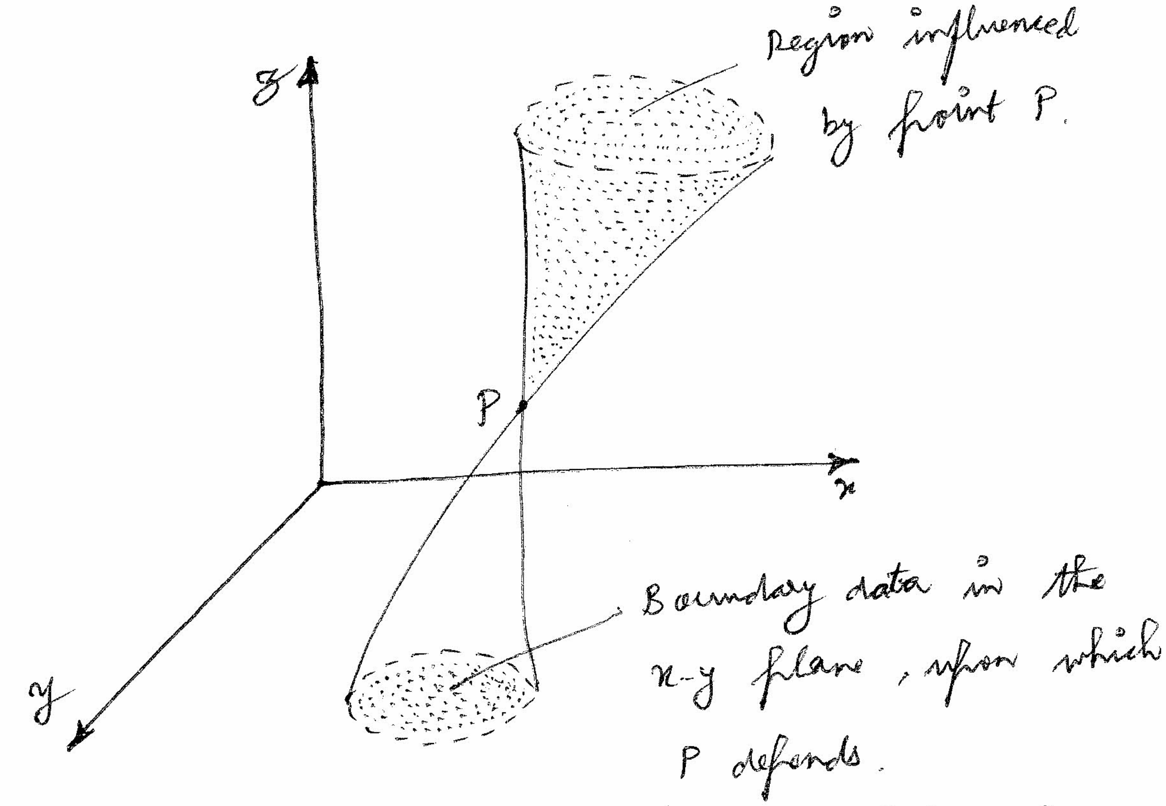
For this reason, point P depends on only that part of the boundary which is intercepted by and included between the two tetreating characteristics lines through point P, in the interval ab.

In fluid dynamics, the following types of flows are governed by hyperbolic partial differential equations.

i) <u>Steady, inviscid supersonic flow.</u>

If the flow is two-dimensional, the behaviour is like discussed from above figure.

If the flow is three-dimensional, there are characteristic surfaces in xyz space.



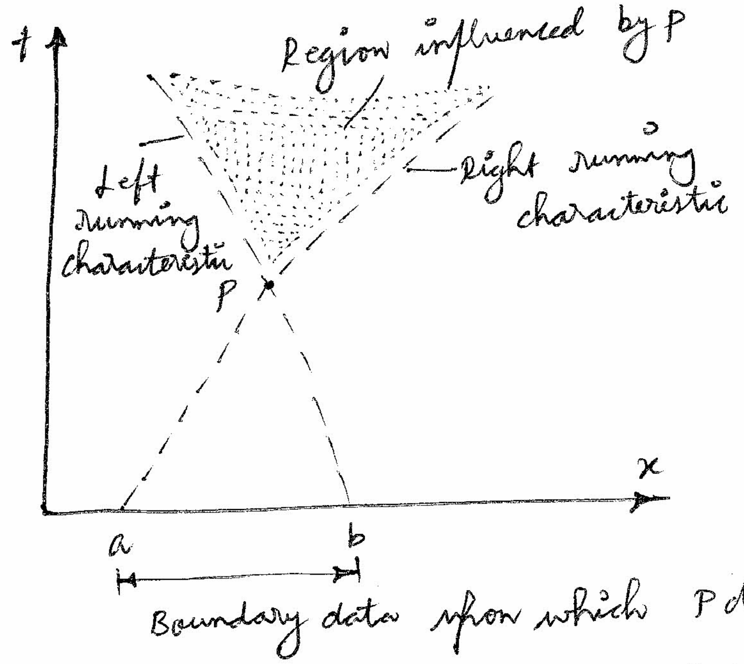

Domain & boundaries for the solution of hyperbolic equations. Three-dimensional steady flow.

Consider point P at a given $(x, y, z)$ location. Information at P influences the shaded volume within the advancing characteristic surface. In addition, if the xy plane ~~p~~ is a boundary surface, then only that portion of the boundary shown as the cross-hatched area in the xy plane, intercepted by the retreating characteristic surface, has any effect on P.

The dependent variables are solved by starting with data given in the xy-plane, and 'marching' in the z-direction. For an inviscid supersonic flow problem, the general flow direction would also be in the z-direction.

ii) <u>Unsteady inviscid compressible flow.</u>

For unsteady one ~~and~~ and two dimensional inviscid flows, the governing equations are hyperbolic, no matter whether the flow is locally subsonic or supersonic. Here time is the marching direction.



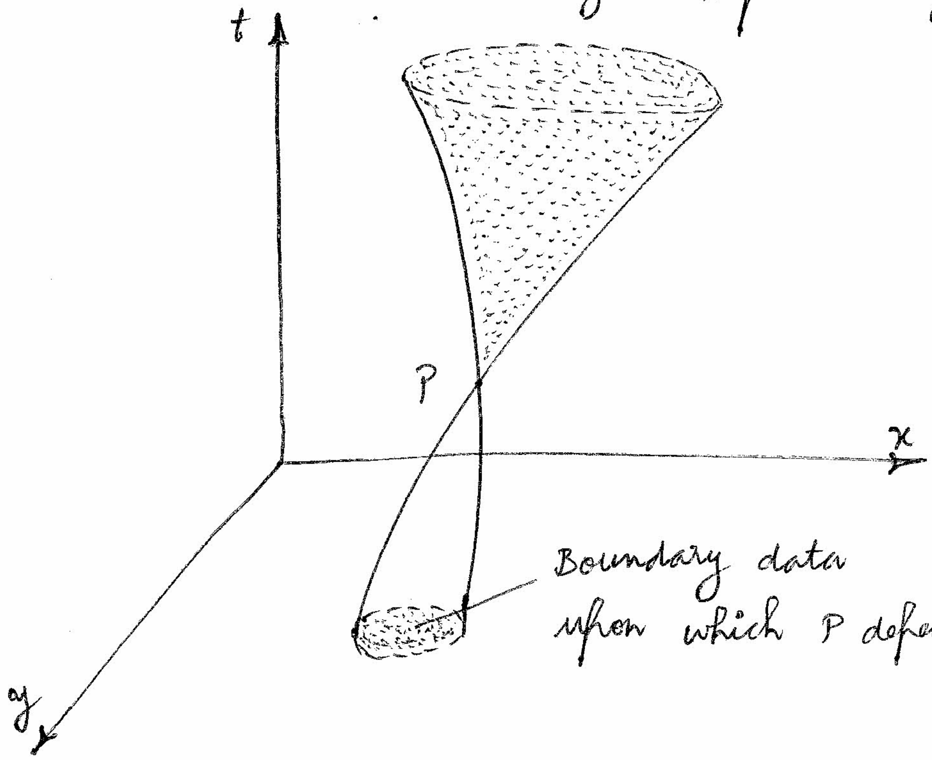

<u>Domain and boundaries for the solution of hyperbolic equations</u>

<u>One dimensional unsteady flow</u>

For one dimensional unsteady flow, consider a point P in the $(x, t)$ plane.

Once again, the region influenced by P is the shaded area between the two advancing characteristics through P, and the interval ab is the only portion of the boundary along the x-axis upon ~~~~ which the solution at P depends.

Region influenced by point P.




## Domain and boundaries for the solution of hyperbolic equation

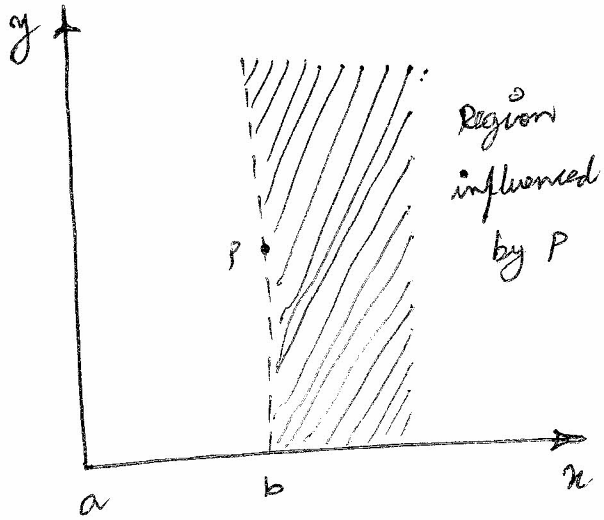
### Two dimensional unsteady flow.

For two dimensional unsteady flow, consider a point in the $(x, y, t)$ space. The region influenced by P, and the portion of the boundary in the xy-plane upon which the solution at P depends. Starting with known initial data in the xy plane, the solution "marches" forward in time.

## Parabolic equations

For parabolic equations, information at point P in the xy-plane influences the entire region of the plane to one side of P.

Let us consider, the single characteristic line passes through point P. Assume the x and y axes are boundaries and the solution at P depends on the boundary conditions along the entire y axis, as well as on that portion of the x-axis from a to b. Solutions to parabolic equations are also 'marching' solutions. IA starting with boundary conditions along both the x and y axes, the flow-field solution is obtained by 'marching' in the general x-direction.




Domain and boundaries for the solution of Parabolic equations in 2D.

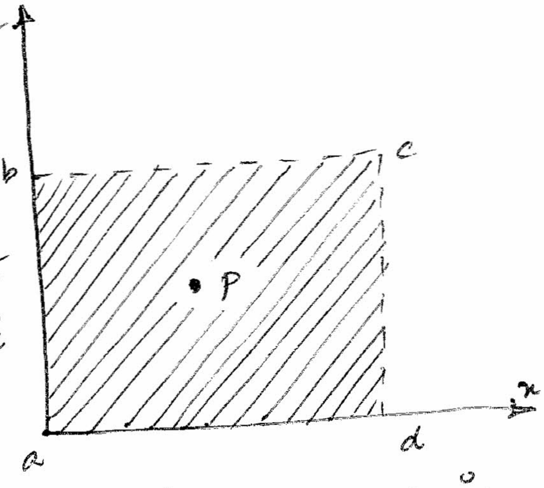
In fluid dynamics, there are reduced forms of the Navier-Stokes equations which exhibit parabolic type behaviour. If the viscous stress terms involving derivatives with respect to x are ignored in these equations, we obtain the 'parabolized' Navier-stokes equations, which allows a solution to march downstream in the x-direction, starting with some prescribed data along the x and y axes.

A further reduction of the Navier-Stokes equations for the case of high Reynolds number leads to the well-known boundary layer equations.

## Elliptic Equations

For elliptic equations, information at point P in the xy-plane influences all other regions of the domain.

The sketch shows a rectangular domain. Here the domain is fully closed, surrounded by the closed boundary abcd. This is in contrast to the open domains for parabolic and hyperbolic equations.




Domain and boundaries for the solution of elliptic equations in 2D

For elliptic equations, because point P influences all points in the domain, then in turn the solution at point P is influenced by the entire closed boundary abcd. Therefore, the solution at point P must be carried out simultaneously with the solution at all other points in the domain. This is in stark contrast to the marching solutions germane to parabolic and hyperbolic equations.

For this reason, problems involving elliptic equations are frequently called 'equilibrium' or 'jury' problems, because the solution within the domain depends on the total boundary around the domain

In fluid dynamics steady, subsonic, inviscid flow is governed by elliptic equations. As a sub-case, this also includes incompressible flow. Hence for such flows, physical boundary conditions must be applied over a closed boundary that totally surrounds the flow, and the flow-field solution at all points in the flow must be obtained simultaneously because the solution at one point influences the solution at all other points. The applied boundary conditions ~~must be~~ can take the following terms.

i) A specification of the dependent variables u and v along the boundary. This type of boundary condition is called the Dirichlet condition.

ii) A specification of derivatives of the dependent variables, such as $\partial u/\partial x$, etc., along the boundary. This type of boundary condition is called the Neumann condition.

## Well - Posed problems

In the solution of partial differential equations it is sometimes easy to attempt a solution using incorrect or insufficient boundary and initial conditions. Whether the solution is being attempted analytically or numerically, such an 'ill-posed' problem will usually lead to spurious results.

Therefore, we define a well-posed problem as follows.

If the solution to a partial differential equation exists and is unique, and if the solution depends continuously upon the initial and boundary conditions, then the problem is well-posed.

## Discretization of Partial differential equations

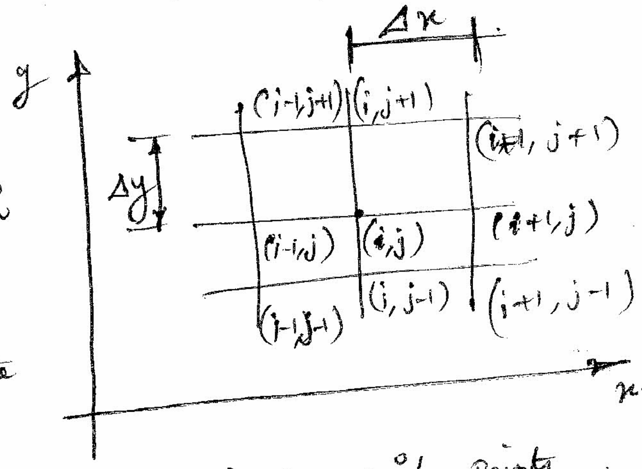
Analytical solutions of partial differential equations involve closed-form expressions which give the variation of the dependent variables continuously throughout the domain.

In contrast, numerical solutions can give answers at only discrete points in the domain, called <u>grid points</u>.

Consider the figure shown, which shows a section of a discrete grid in the $xy$-plane. Let us assume that the spacing of the grid points in the $x$ & $y$ directions are uniform and are given by $\Delta x$ & $\Delta y$.

In general, $\Delta x$ and $\Delta y$ are different. Indeed, it is not absolutely necessary that $\Delta x$ or $\Delta y$ be uniform. We will assume $\Delta x$ and $\Delta y$ to be constants, but that $\Delta x$ does not have to equal $\Delta y$.

The grid points are identified by an index $i$ which runs in the $x$-direction and an index $j$ which runs in the $y$-direction.




Discrete Grid points

The method of finite-differences is widely used in CFD. The philosophy of finite difference methods is to replace the partial derivatives appearing in the governing equations of fluid dynamics with algebraic difference quotients, yielding a system of algebraic equations which can be solved for the flow-field variables at the specific, discrete grid points in the flow.

## Derivation of Elementary finite difference quotients

Finite difference representations of derivatives are based on Taylor's series expansions. For example, if $u_{i,j}$ denotes the $x$-component of velocity at point $(i,j)$, then the velocity $u_{i+1,j}$ at point $(i+1,j)$ can be expressed in terms of a Taylor's series expanded about point $(i,j)$ as follows.

$$u_{i+1,j} = u_{i,j} + \left(\frac{\partial u}{\partial x}\right)_{i,j} \Delta x + \left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} \frac{\Delta x^2}{2} + \left(\frac{\partial^3 u}{\partial x^3}\right)_{i,j} \frac{\Delta x^3}{6} + \cdots \quad \to (1)$$

Eqn (1) is mathematically an exact expression for $u_{i+1,j}$ if,

i) The number of terms is infinite and the series converges

ii) and/or $\Delta x \to 0$.

For numerical computations, it is impractical to carry an infinite number of terms in eqn (1). Therefore eqn (1) is truncated. For example, if terms of magnitude $(\Delta x)^3$ and higher order are neglected, Eqn ① reduces to

$$u_{i+1,j} \simeq u_{i,j} + \left(\frac{\partial u}{\partial x}\right)_{i,j} \Delta x + \left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} \frac{(\Delta x)^2}{2} \quad \rightarrow ②$$

Eqn ② is of second order accuracy, because terms of order $(\Delta x)^3$ and higher have been neglected. If terms of order $(\Delta x)^2$ and higher are neglected.

$$u_{i+1,j} \simeq u_{i,j} + \left(\frac{\partial u}{\partial x}\right)_{i,j} \Delta x. \quad \longrightarrow ③$$

Eqn ③ is of first order accuracy. The neglected higher-order terms represent the truncation error in the finite series representation. The truncation error can be reduced by:

i) carrying more terms in the Taylor's series. This leads to higher-order accuracy in the representation of $u_{i+1,j}$.

ii) Reducing the magnitude of $\Delta x$.

Let us solve eqn ① for $\left(\frac{\partial u}{\partial x}\right)_{i,j}$.

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} \Delta x = u_{i+1,j} - u_{i,j} - \left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} \frac{(\Delta x)^2}{2} - \left(\frac{\partial^3 u}{\partial x^3}\right)_{i,j} \frac{\Delta x^3}{6} - \cdots$$

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i+1,j} - u_{i,j}}{\Delta x} \underbrace{- \left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} \frac{\Delta x}{2} - \left(\frac{\partial^3 u}{\partial x^3}\right)_{i,j} \frac{\Delta x^2}{6} \cdots}_{\text{Truncation error}}$$

or

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i+1,j} - u_{i,j}}{\Delta x} + O(\Delta x) \rightarrow (4)$$

In eqn (4) the symbol $O(\Delta x)$ is a formal mathematical notation which represents 'terms of-order-of $\Delta x$'. And this is the first-order forward difference for accurate difference representation for the derivative $\left(\frac{\partial u}{\partial x}\right)_{i,j}$.

Let us now write a Taylor's series expansion for $u_{i-1,j}$, expanded about $u_{i,j}$.

$$u_{i-1,j} = u_{i,j} + \left(\frac{\partial u}{\partial x}\right)_{i,j} (-\Delta x) + \left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} \frac{(-\Delta x)^2}{2} + \left(\frac{\partial^3 u}{\partial x^3}\right)_{i,j} \frac{(-\Delta x)^3}{6} + \cdots \rightarrow (5)$$

Solving for $\left(\frac{\partial u}{\partial x}\right)_{i,j}$, we obtain

$$-\left(\frac{\partial u}{\partial x}\right)_{i,j} \Delta x = u_{i-1,j} - u_{i,j} - \left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} \frac{(-\Delta x)^2}{2} - \left(\frac{\partial^3 u}{\partial x^3}\right)_{i,j} \frac{(-\Delta x)^3}{6} + \cdots$$

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i,j} - u_{i-1,j}}{\Delta x} + \left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} \frac{\Delta x}{2} - \left(\frac{\partial^3 u}{\partial x^3}\right)_{i,j} \frac{\Delta x^2}{6} + \cdots$$

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i,j} - u_{i-1,j}}{\Delta x} + O(\Delta x) \rightarrow (6)$$

Eqn (6) is a first order rearward difference expression for the derivative $\left(\frac{\partial u}{\partial x}\right)$ at grid point $(i,j)$.

Let us now subtract eqn (5) from (1)

$$u_{i+1,j} - u_{i-1,j} = \cancel{u_{i,j}} + \left(\frac{\partial u}{\partial x}\right)_{i,j} \Delta x + \cancel{\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} \frac{(\Delta x)^2}{2}} + \left(\frac{\partial^3 u}{\partial x^3}\right)_{i,j} \frac{(\Delta x)^3}{6} + \cdots$$

$$- \cancel{u_{i,j}} + \left(\frac{\partial u}{\partial x}\right)_{i,j} \Delta x - \cancel{\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} \frac{\Delta x^2}{2}} + \left(\frac{\partial^3 u}{\partial x^3}\right)_{i,j} \frac{\Delta x^3}{6}$$

$$= 2\left(\frac{\partial u}{\partial x}\right)_{i,j} \Delta x + \left(\frac{\partial^3 u}{\partial x^3}\right)_{i,j} \frac{\Delta x^3}{3} + \cdots \rightarrow (7)$$

Solving for $\left(\frac{\partial u}{\partial x}\right)_{i,j}$, we obtain

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i+1,j} - u_{i-1,j}}{2\,\Delta x} + O(\Delta x)^2 \rightarrow (8)$$

Eqn (8) is a second order central difference for the derivative $\left(\frac{\partial u}{\partial x}\right)$ at grid point $(i,j)$

To obtain a finite-difference expression for the second partial derivative $\left(\partial^2 u/\partial x^2\right)_{i,j}$, from eqn (8)

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i+1,j} - u_{i-1,j}}{2\,\Delta x} - \left(\frac{\partial^3 u}{\partial x^3}\right)_{i,j} \frac{(\Delta x)^2}{6} \rightarrow (9)$$

Substitute eqn (9) in (1).

$$u_{i+1,j} = u_{i,j} + \left[\frac{u_{i+1,j}-u_{i-1,j}}{2\Delta x} - \left(\frac{\partial^3 u}{\partial x^3}\right)_{i,j}\frac{(\Delta x)^2}{6} + \cdots\right]\Delta x$$

$$+\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j}\frac{(\Delta x)^2}{2} + \left(\frac{\partial^3 u}{\partial x^3}\right)_{i,j}\frac{(\Delta x)^3}{6} + \left(\frac{\partial^4 u}{\partial x^4}\right)_{i,j}\frac{(\Delta x)^4}{24} + \cdots$$

Solving for $\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j}$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j}\frac{(\Delta x)^2}{2} = u_{i+1,j} - u_{i,j} - \left[\frac{u_{i+1,j}-u_{i-1,j}}{2}\right]$$

$$+\cancel{\left(\frac{\partial^3 u}{\partial x^3}\right)_{i,j}\frac{(\Delta x)^3}{6}} - \cancel{\left(\frac{\partial^3 u}{\partial x^3}\right)_{i,j}\frac{(\Delta x)^3}{6}}$$

$$-\left(\frac{\partial^4 u}{\partial x^4}\right)_{i,j}\frac{(\Delta x)^4}{24}$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = \frac{2u_{i+1,j} - 2u_{i,j} - u_{i+1,j} + u_{i-1,j}}{\cancel{2}}\,\frac{\cancel{2}}{(\Delta x)^2}$$

$$-\left(\frac{\partial^4 u}{\partial x^4}\right)_{i,j}\frac{(\Delta x)^{\cancel{4}\,2}}{\cancel{24}\,12}\times\frac{\cancel{2}}{\cancel{(\Delta x)^2}}$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = \left[\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2}\right] + O(\Delta x)^2 \quad \rightarrow (10)$$

Eqn (10) is a second-order central second difference for the derivative $\left(\frac{\partial^2 u}{\partial x^2}\right)$ at grid point $(i,j)$.

|||ly the difference expressions for the y-directions are,

First order forward difference

$$\left(\frac{\partial u}{\partial y}\right)_{i,j} = \frac{u_{i,j+1} - u_{i,j}}{\Delta y} + O(\Delta y)$$

First order rearward difference.

$$\left(\frac{\partial u}{\partial y}\right)_{i,j} = \frac{u_{i,j} - u_{i,j-1}}{\Delta y} + O(\Delta y)$$

Second order central difference

$$\left(\frac{\partial u}{\partial y}\right)_{i,j} = \frac{u_{i,j+1} - u_{i,j-1}}{2\Delta y} + O(\Delta y)^2$$

Second order central second difference

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{i,j} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(\Delta y)^2} + O(\Delta y)^2$$

Note that the central second difference can be intepreted as a forward difference of the first derivating, with rearward derivatives used for the first derivatives.

Dropping the O notation, we have

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{ij} = \left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\right]_{i,j}$$

$$\simeq \frac{\left(\frac{\partial u}{\partial x}\right)_{i+1,j} - \left(\frac{\partial u}{\partial x}\right)_{i,j}}{\Delta x}$$

$$\simeq \left[\left(\frac{u_{i+1,j} - u_{i,j}}{\Delta x}\right) - \left(\frac{u_{i,j} - u_{i-1,j}}{\Delta x}\right)\right]\frac{1}{\Delta x}$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} \simeq \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2} \longrightarrow (11)$$

This is the same difference quotient as eqn (10). The finite difference quotient for the mixed derivative $\left(\frac{\partial^2 u}{\partial x \partial y}\right)$ at grid point $(i,j)$ can be derived as.

$$\left(\frac{\partial^2 u}{\partial x \partial y}\right)_{i,j} = \left[\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)\right]_{i,j} \simeq \frac{\left(\frac{\partial u}{\partial y}\right)_{i+1,j} - \left(\frac{\partial u}{\partial y}\right)_{i-1,j}}{2\,\Delta x}$$

$$\simeq \left[\frac{u_{i+1,j+1} - u_{i+1,j-1}}{2\,\Delta y}\right] - \left[\frac{u_{i-1,j+1} - u_{i-1,j-1}}{2\,\Delta y}\right]\frac{1}{2\Delta x}$$

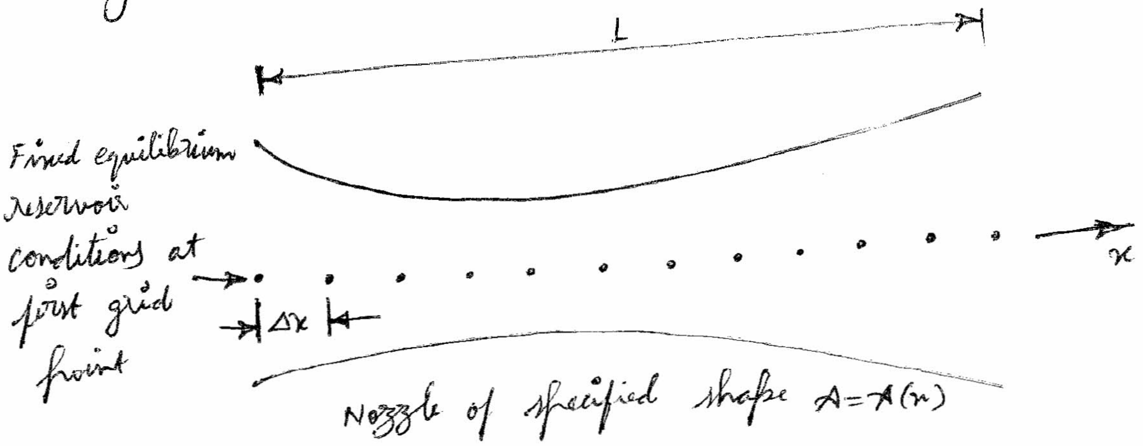
$$\therefore \left(\frac{\partial^2 u}{\partial x \partial y}\right)_{i,j} \simeq \frac{1}{4\Delta x \Delta y}\left[u_{i+1,j+1} - u_{i+1,j-1} - u_{i-1,j+1} + u_{i-1,j-1}\right] \longrightarrow (12)$$

These are the forms of finite difference quotients

## Explicit Finite Difference Methods.

### The Lax-Wendroff Method.

Let us consider the subsonic-supersonic flow through a convergent-divergent (CD) Nozzle.




Flow through a convergent-divergent nozzle.

Here, a nozzle of specified area distribution, $A=A(x)$, is given and the reservoir conditions are known. The nozzle is divided into a number of grids points in the $x$-direction. The spacing between adjacent grid points is $\Delta x$. Now assume values of the flow field variables at all grid points and consider this rather arbitrarily assumed flow as an initial condition at time $t=0$.

Consider a grid point, say point $i$. Let $g_i$ denote a flow field variable at this point.

This variable $g_i$ will be a function of time. Now calculate a new value of $g_i$ at time $t+\Delta t$. Starting from the initial conditions, the first new time is

$$t + \Delta t = 0 + \Delta t.$$

The new value of $g_i$ is obtained from a Taylor's series expansion in time

$$g_i(t+\Delta t) = g_i(t) + \left(\frac{\partial g}{\partial t}\right)_i \Delta t + \left(\frac{\partial^2 g}{\partial t^2}\right)_i \frac{(\Delta t)^2}{2!} + \cdots$$

or using the standard notation of time

$$g_i^{(t+\Delta t)} = g_i^t + \left(\frac{\partial g}{\partial t}\right)_i^t \Delta t + \left(\frac{\partial^2 g}{\partial t^2}\right)_i^t \frac{(\Delta t)^2}{2} + \cdots \longrightarrow (1)$$

$g_i^t$ is known and $\Delta t$ is specified. If we have values for the derivatives $\left(\frac{\partial g}{\partial t}\right)_i^t$ and $\left(\frac{\partial^2 g}{\partial t^2}\right)_i^t$ the $g_i^{t+\Delta t}$ can be calculated. The values for the derivatives are obtained from the governing flow equations. The governing flow equations for the quasi-one-dimensional flow through a nozzle are

continuity : $\dfrac{\partial \rho}{\partial t} = -\dfrac{1}{A}\,\dfrac{\partial(\rho u A)}{\partial x}$

Momentum : $$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\left(\frac{\partial p}{\partial x} + \rho u \frac{\partial u}{\partial x}\right)$$

Energy : $$\frac{\partial e}{\partial t} = -\frac{1}{\rho}\left[p\frac{\partial u}{\partial x} + pu\frac{\partial(\ln A)}{\partial x} + \rho u \frac{\partial e}{\partial x}\right]$$

These eqns are written with the time derivatives on the left-hand side and spatial derivatives on the right-hand side. Let us consider the continuity eqn.

$$\frac{\partial \rho}{\partial t} = -\frac{1}{A}\rho u \frac{\partial A}{\partial x} - \frac{1}{\cancel{A}}\rho \cancel{A}\frac{\partial u}{\partial x} - \frac{1}{\cancel{A}} u \cancel{A}\frac{\partial \rho}{\partial x}$$

$$\frac{\partial \rho}{\partial t} = -\frac{1}{A}\rho u \frac{\partial A}{\partial x} - \rho\frac{\partial u}{\partial x} - u\frac{\partial \rho}{\partial x} \longrightarrow (2)$$

At time $t=0$, the flow field variables are assumed. Let us replace the spatial derivatives with central difference.

$$\left(\frac{\partial \rho}{\partial t}\right)_i^t = -\frac{1}{A}\rho_i^t u_i^t\left(\frac{A_{i+1} - A_{i-1}}{2\Delta x}\right) - \rho_i^t\left(\frac{u_{i+1}^t - u_{i-1}^t}{2\Delta x}\right) - u_i^t\left(\frac{\rho_{i+1}^t - \rho_{i-1}^t}{2\Delta x}\right) \longrightarrow (3)$$

For second partial differentiate the continuity eqn w.r.t. time.

$$\frac{\partial^2 \rho}{\partial t^2} = -\frac{1}{A}\frac{\partial}{\partial t}\left(\rho u \frac{\partial A}{\partial x}\right) - \frac{\partial}{\partial t}\left(\rho\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial t}\left(u\frac{\partial \rho}{\partial x}\right)$$

$$= -\frac{1}{A}\left[\frac{\partial A}{\partial x}\left(\rho\frac{\partial u}{\partial t} + u\frac{\partial \rho}{\partial t}\right)\right] - \cancel{\frac{\partial \rho}{\partial t}\rho\frac{\partial^2 u}{}}\left(\frac{\partial \rho}{\partial t}\right)\left(\frac{\partial u}{\partial x}\right)$$

$$-\rho\frac{\partial^2 u}{\partial x \partial t} - \left(\frac{\partial u}{\partial t}\right)\left(\frac{\partial \rho}{\partial x}\right) - u\frac{\partial^2 \rho}{\partial x \partial t} \longrightarrow (4)$$

To find $\frac{\partial^2 \rho}{\partial t \partial x}$, the continuity eqn is differentiated w.r.t. $x$

$$\frac{\partial^2 \rho}{\partial x \partial t} = -\frac{1}{A}\frac{\partial}{\partial x}\left(\rho u \frac{\partial A}{\partial x}\right) - \frac{\partial}{\partial x}\left(\rho \frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial x}\left(u \frac{\partial \rho}{\partial x}\right)$$

$$= -\frac{1}{A}\left[\frac{\partial A}{\partial x}\left(\rho \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial x}\right) + \rho u \frac{\partial^2 A}{\partial x^2}\right] - \left(\frac{\partial \rho}{\partial x}\right)\left(\frac{\partial u}{\partial x}\right)$$

$$- \rho \frac{\partial^2 u}{\partial x^2} - \left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial \rho}{\partial x}\right) - u \frac{\partial^2 \rho}{\partial x^2} \longrightarrow (5)$$

Where,

$$\frac{\partial^2 A}{\partial x^2} = \frac{A^*_{i+1} - 2A^*_i + A^*_{i-1}}{(\Delta x)^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u^t_{i+1} - 2u^t_i + u^t_{i-1}}{(\Delta x)^2}$$

$$\frac{\partial^2 \rho}{\partial x^2} = \frac{\rho^t_{i+1} - 2\rho^t_i + \rho^t_{i-1}}{(\Delta x)^2}$$

To find $\frac{\partial^2 u}{\partial x \partial t}$, the momentum eqn is differentiated w.r.t. $x$.

$$\frac{\partial^2 u}{\partial x \partial t} = -\frac{1}{\rho}\left[\frac{\partial^2 p}{\partial x^2} + \rho u \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\left(\rho \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial x}\right)\right] \longrightarrow (6)$$

where, $\boxed{\left(\frac{\partial^2 p}{\partial x^2}\right) = \frac{p^t_{i+1} - 2p^t_i + p^t_{i-1}}{(\Delta x)^2}}$

Every quantity on the right hand side of eqn (1) is now known. This allows the density $\rho_i^{t+\Delta t}$ to be calculated.

## MacCormack's Method.

It is the most popular explicit finite-difference method for solving fluid flows. It is closely related to the Lax-Wendroff method, but is easier to apply. Let us use the same CD nozzle problem ~~discussed~~ to illustrate MacCormack's method. It is also based on a Taylor's series expansion in time.

Let us consider the density at grid point i.

$$\rho_i^{t+\Delta t} = \rho_i^{t} + \left(\frac{\partial \rho}{\partial t}\right)_{ave} \Delta t \rightarrow (1)$$

Eqn (1) is a truncated Taylor's series and is first order accurate. $\left(\frac{\partial \rho}{\partial t}\right)_{ave}$ is an average time derivative taken between time t and $t+\Delta t$. The average time derivative in eqn (1) is evaluated from a predictor-corrector philosophy as follows,

### Predictor step

From continuity equation,

$$\frac{\partial \rho}{\partial t} = -\frac{1}{A}\rho u \frac{\partial A}{\partial x} - \rho\frac{\partial u}{\partial x} - u\frac{\partial \rho}{\partial x} \rightarrow (2)$$

In equation (2) calculate the spatial derivatives from the known flow field values at time $t$ using forward differences.

$$\left(\frac{\partial \rho}{\partial t}\right)_i^t = -\frac{1}{A}\rho_i^t u_i^t\left(\frac{A_{i+1}-A_i}{\Delta x}\right) - \rho_i^t\left(\frac{u_{i+1}^t - u_i^t}{\Delta x}\right) - u_i^t\left(\frac{\rho_{i+1}^t - \rho_i^t}{\Delta x}\right) \rightarrow (3)$$

obtain a predicted value of density, $\bar{\rho}_i^{(t+\Delta t)}$ from the first two terms of a Taylor's series as follows

$$\bar{\rho}_i^{(t+\Delta t)} = \rho_i^t + \left(\frac{\partial \rho}{\partial t}\right)_i^t \Delta t \rightarrow (4)$$

In eqn (4), $\rho_i^t$ is known and $\left(\frac{\partial \rho}{\partial t}\right)_i^t$ is a known value from eqn (3) hence, $\bar{\rho}_i^{(t+\Delta t)}$ is obtained.

In a similar fashion, from the momentum and energy equations, predicted values of the other flow variables such as $\bar{u}_i^{(t+\Delta t)}$, $\bar{e}_i^{(t+\Delta t)}$, etc... are obtained.

<u>corrector step</u>

Here, we first obtain a predicted value of the time derivative, $\left(\overline{\frac{\partial \rho}{\partial t}}\right)_i^{t+\Delta t}$, by substituting the predicted values using rearward differences.

$$\left(\overline{\frac{\partial \rho}{\partial t}}\right)_i^{(t+\Delta t)} = -\frac{1}{A}\,\bar{\rho}_i^{(t+\Delta t)}\,\bar{u}_i^{(t+\Delta t)}\left(\frac{A_i - A_{i-1}}{\Delta x}\right) - \bar{\rho}_i^{(t+\Delta t)}\left(\frac{\bar{u}_i^{(t+\Delta t)} - \bar{u}_{i-1}^{(t+\Delta t)}}{\Delta x}\right)$$

$$-\,\bar{u}_i^{(t+\Delta t)}\left(\frac{\bar{\rho}_i^{(t+\Delta t)} - \bar{\rho}_{i-1}^{(t+\Delta t)}}{\Delta x}\right) \longrightarrow (5)$$

Now calculate the average time derivative as the arithmetic mean between eqns (3) & (5).

$$\left(\frac{\partial \rho}{\partial t}\right)_{ave} = \frac{1}{2}\left[\left(\frac{\partial \rho}{\partial t}\right)_i^{t} + \left(\overline{\frac{\partial \rho}{\partial t}}\right)_i^{(t+\Delta t)}\right]. \quad (6)$$

by substituting eqn (6) in (1) we obtain the corrected value of $\rho_i^{t+\Delta t}$.

The above predictor-corrector approach is carried out for all grid points throughout the nozzle, and is applied simultaneously to the momentum and energy equations in order to generate $u_i^{t+\Delta t}$ and $e_i^{t+\Delta t}$. In this fashion, the flow field through the entire nozzle at time $t+\Delta t$ is calculated. This is repeated for a large number of time steps until the steady state is achieved.

# UNIT - II GRID GENERATION



## General transformation of the Equations

Consider a two-dimensional unsteady flow, with independent variables $x, y$ and $t$. Transform the variables from physical space $(x, y, t)$ to a transformed space $(\xi, \eta, \tau)$, where

$$\xi = \xi(x, y, t)$$

$$\eta = \eta(x, y, t)$$

$$\tau = \tau(t)$$

In the above transformation, $\tau$ is considered a function of $t$ only and is frequently given by $\tau = t$. From the chain rule of differential calculus, we have

$$\left(\frac{\partial}{\partial x}\right)_{y,t} = \left(\frac{\partial}{\partial \xi}\right)_{\eta,\tau}\left(\frac{\partial \xi}{\partial x}\right)_{y,t} + \left(\frac{\partial}{\partial \eta}\right)_{\xi,\tau}\left(\frac{\partial \eta}{\partial x}\right)_{y,t} + \left(\frac{\partial}{\partial \tau}\right)_{\xi,\eta}\cancel{\left(\frac{\partial \tau}{\partial x}\right)_{y,t}}^{\,0}$$

generally

$$\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial \xi}{\partial x}\right) + \left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial \eta}{\partial x}\right) \rightarrow (1)$$

Similarly

$$\frac{\partial}{\partial y} = \left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial \xi}{\partial y}\right) + \left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial \eta}{\partial y}\right) \rightarrow (2)$$

Also,

$$\left(\frac{\partial}{\partial t}\right)_{x,y} = \left(\frac{\partial}{\partial \xi}\right)_{\eta,\tau}\left(\frac{\partial \xi}{\partial t}\right)_{x,y} + \left(\frac{\partial}{\partial \eta}\right)_{\xi,\tau}\left(\frac{\partial \eta}{\partial t}\right)_{x,y}$$

$$+ \left(\frac{\partial}{\partial \tau}\right)_{\xi,\eta}\left(\frac{\partial \tau}{\partial t}\right)_{x,y}$$

or

$$\frac{\partial}{\partial t} = \left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial \xi}{\partial t}\right) + \left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial \eta}{\partial t}\right) + \left(\frac{\partial}{\partial \tau}\right)\left(\frac{\partial \tau}{\partial t}\right) \quad (3)$$

Eqn (1), (2) & (3) allow the derivatives w.r.t $x, y$ & $t$ to be transformed into derivatives with respect to $\xi, \eta$ & $\tau$. The coefficients of the derivatives with respect to $\xi, \eta$ & $\tau$ are called metrics. The transformation is frequently a purely numerical relationship, in which the metrics can be evaluated by finite-difference quotients - typically central differences.

The equations for viscous flow involve second derivatives. Therefore, we need a transformation for these derivatives and those can be obtained as follows.

$$A = \frac{\partial}{\partial x} = \left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial \xi}{\partial x}\right) + \left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial \eta}{\partial x}\right)$$

Then

$$\frac{\partial A}{\partial x} = \frac{\partial^2}{\partial x^2} = \frac{\partial}{\partial x}\left[\left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial \xi}{\partial x}\right) + \left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial \eta}{\partial x}\right)\right]$$

$$\frac{\partial^2}{\partial x^2} = \left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial^2 \xi}{\partial x^2}\right) + \left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial^2}{\partial x \partial \xi}\right) + \left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial^2 \eta}{\partial x^2}\right) + \left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial^2}{\partial x \partial \eta}\right) \quad \to (4)$$

The mixed derivatives in eqn (4) are denoted by B & C can be obtained from the chain rule as follows.

$$B = \frac{\partial^2}{\partial x \partial \xi} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial \xi}\right)$$

$$= \left(\frac{\partial^2}{\partial \xi^2}\right)\left(\frac{\partial \xi}{\partial x}\right) + \left(\frac{\partial^2}{\partial \eta \partial \xi}\right)\left(\frac{\partial \eta}{\partial x}\right)$$

Similarly,

$$C = \frac{\partial^2}{\partial x \partial \eta} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial \eta}\right)$$

$$= \left(\frac{\partial^2}{\partial \xi \partial \eta}\right)\left(\frac{\partial \xi}{\partial x}\right) + \left(\frac{\partial^2}{\partial \eta^2}\right)\left(\frac{\partial \eta}{\partial x}\right)$$

Substitute eqn B & C in (4)

$$\frac{\partial^2}{\partial x^2} = \left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial^2 \xi}{\partial x^2}\right) + \left(\frac{\partial \xi}{\partial x}\right)\left[\left(\frac{\partial^2}{\partial \xi^2}\right)\left(\frac{\partial \xi}{\partial x}\right) + \left(\frac{\partial^2}{\partial \eta \partial \xi}\right)\left(\frac{\partial \eta}{\partial x}\right)\right]$$
$$+ \left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial^2 \eta}{\partial x^2}\right) + \left(\frac{\partial \eta}{\partial x}\right)\left[\left(\frac{\partial^2}{\partial \xi \partial \eta}\right)\left(\frac{\partial \xi}{\partial x}\right) + \left(\frac{\partial^2}{\partial \eta^2}\right)\left(\frac{\partial \eta}{\partial x}\right)\right] \quad \to (5)$$

Short it like eqn (7)

Eqn (5) gives the second partial derivative with respect to $x$ in terms of first, second and mixed derivatives with respect to $\xi$ and $\eta$, multiplied by various metric terms.

Let us now continue to obtain the second partial with respect to $y$.

$$D = \frac{\partial}{\partial y} = \left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial \xi}{\partial y}\right) + \left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial \eta}{\partial y}\right)$$

Then,

$$\frac{\partial D}{\partial y} = \frac{\partial^2}{\partial y^2} = \frac{\partial}{\partial y}\left[\left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial \xi}{\partial y}\right) + \left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial \eta}{\partial y}\right)\right]$$

$$= \left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial^2 \xi}{\partial y^2}\right) + \left(\frac{\partial \xi}{\partial y}\right)\left(\frac{\partial^2}{\partial \xi \partial y}\right)$$

$$+ \left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial^2 \eta}{\partial y^2}\right) + \left(\frac{\partial \eta}{\partial y}\right)\left(\frac{\partial^2}{\partial \eta \partial y}\right) \longrightarrow (6)$$

The mixed derivatives in eqn (6) are denoted by E & F can be obtained as follows.

$$E = \left(\frac{\partial^2}{\partial y \partial \xi}\right) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial \xi}\right)$$

$$= \left(\frac{\partial^2}{\partial \xi^2}\right)\left(\frac{\partial \xi}{\partial y}\right) + \left(\frac{\partial^2}{\partial \xi \partial \eta}\right)\left(\frac{\partial \eta}{\partial y}\right)$$

and

$$F = \left(\frac{\partial^2}{\partial y \partial \eta}\right) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial \eta}\right)$$

$$= \left(\frac{\partial^2}{\partial \xi \partial \eta}\right)\left(\frac{\partial \xi}{\partial y}\right) + \left(\frac{\partial^2}{\partial \eta^2}\right)\left(\frac{\partial \eta}{\partial y}\right)$$

substitute eqn E & F in (6)

$$\frac{\partial^2}{\partial y^2} = \left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial^2 \xi}{\partial y^2}\right) + \left(\frac{\partial \xi}{\partial y}\right)\left[\left(\frac{\partial^2}{\partial \xi^2}\right)\left(\frac{\partial \xi}{\partial y}\right) + \left(\frac{\partial^2}{\partial \xi \partial \eta}\right)\left(\frac{\partial \eta}{\partial y}\right)\right]$$

$$+\left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial^2 \eta}{\partial y^2}\right) + \left(\frac{\partial \eta}{\partial y}\right)\left[\left(\frac{\partial^2}{\partial \xi \partial \eta}\right)\left(\frac{\partial \xi}{\partial y}\right) + \left(\frac{\partial^2}{\partial \eta^2}\right)\left(\frac{\partial \eta}{\partial y}\right)\right]$$

$$= \left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial^2 \xi}{\partial y^2}\right) + \left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial^2 \eta}{\partial y^2}\right) + \left(\frac{\partial^2}{\partial \xi^2}\right)\left(\frac{\partial \xi}{\partial y}\right)^2$$

$$+ \left(\frac{\partial^2}{\partial \eta^2}\right)\left(\frac{\partial \eta}{\partial y}\right)^2 + 2\left(\frac{\partial^2}{\partial \xi \partial \eta}\right)\left(\frac{\partial \xi}{\partial y}\right)\left(\frac{\partial \eta}{\partial y}\right) \quad \to (7)$$

Eqn (7) gives the second partial derivative with respect to y in terms of first, second and mixed derivatives with respect to $\xi$ and $\eta$ multiplied by various metric terms with respect to x and y.

To obtain the second partial with respect to x and y.

$$\frac{\partial^2}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\right) = \frac{\partial D}{\partial x} = \frac{\partial}{\partial x}\left[\left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial \xi}{\partial y}\right) + \left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial \eta}{\partial y}\right)\right]$$

$$\frac{\partial^2}{\partial x \partial y} = \left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial^2 \xi}{\partial x \partial y}\right) + \left(\frac{\partial \xi}{\partial y}\right)\left(\frac{\partial^2}{\partial x \partial \xi}\right) + \left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial^2 \eta}{\partial x \partial y}\right)$$

$$+\left(\frac{\partial \eta}{\partial y}\right)\left(\frac{\partial^2}{\partial x \partial \eta}\right) \quad \to (8)$$

Substitute B & C in eqn (8)

$$\frac{\partial^2}{\partial x \partial y} = \left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial^2 \xi}{\partial x \partial y}\right) + \left(\frac{\partial \xi}{\partial y}\right)\left[\left(\frac{\partial^2}{\partial \xi^2}\right)\left(\frac{\partial \xi}{\partial x}\right) + \left(\frac{\partial^2}{\partial \eta \partial \xi}\right)\left(\frac{\partial \eta}{\partial x}\right)\right]$$

$$+\left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial^2 \eta}{\partial x \partial y}\right) + \left(\frac{\partial \eta}{\partial y}\right)\left[\left(\frac{\partial^2}{\partial \xi \partial \eta}\right)\left(\frac{\partial \xi}{\partial x}\right) + \left(\frac{\partial^2}{\partial \eta^2}\right)\left(\frac{\partial \eta}{\partial x}\right)\right]$$

$$= \left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial^2 \xi}{\partial x \partial y}\right) + \left(\frac{\partial}{\partial \eta}\right)\left(\frac{\partial^2 \eta}{\partial x \partial y}\right) + \left(\frac{\partial^2}{\partial \xi^2}\right)\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \xi}{\partial y}\right)$$

$$+ \left(\frac{\partial^2}{\partial \eta^2}\right)\left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial \eta}{\partial y}\right) + \left(\frac{\partial^2}{\partial \eta \partial \xi}\right)\left[\left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial \xi}{\partial y}\right) + \left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \eta}{\partial y}\right)\right] \rightarrow (9)$$

Eqn (9) gives the second partial derivative with respect to $x$ and $y$ in terms of first, second and mixed derivatives with respect to $\xi$ and $\eta$, multiplied by various metric terms.

When this transformation is made, the governing equations in terms of $\xi$, $\eta$ and $\tau$ become lengthy. Let us consider a simple example, namely that for inviscid, irrotational, steady, incompressible flow, for which Laplace's Equation is the governing equation.

$$\text{Laplace's Equation}: \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \rightarrow (10)$$

It is in physical $(x, y)$ space.

Substitute (5) & (7) in (10).

$$\left(\frac{\partial \phi}{\partial \xi}\right)\left(\frac{\partial^2 \xi}{\partial x^2}\right) + \left(\frac{\partial^2 \phi}{\partial \xi^2}\right)\left(\frac{\partial \xi}{\partial x}\right)^2 + 2\left(\frac{\partial^2 \phi}{\partial \eta \partial \xi}\right)\left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial \xi}{\partial x}\right)$$

$$+\left(\frac{\partial \phi}{\partial \eta}\right)\left(\frac{\partial^2 \eta}{\partial x^2}\right) + \cancel{\frac{\partial^2 \phi}{\partial \xi \partial \eta}} + \left(\frac{\partial^2 \phi}{\partial \eta^2}\right)\left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial \xi}\right)\left(\frac{\partial^2 \xi}{\partial y^2}\right)$$

$$+\left(\frac{\partial^2 \phi}{\partial \xi^2}\right)\left(\frac{\partial \xi}{\partial y}\right)^2 + 2\left(\frac{\partial^2 \phi}{\partial \xi \partial \eta}\right)\left(\frac{\partial \eta}{\partial y}\right)\left(\frac{\partial \xi}{\partial y}\right)$$

$$+\left(\frac{\partial \phi}{\partial \eta}\right)\left(\frac{\partial^2 \eta}{\partial y^2}\right) + \left(\frac{\partial^2 \phi}{\partial \eta^2}\right)\left(\frac{\partial \eta}{\partial y}\right)^2 = 0$$

$$\left(\frac{\partial^2 \phi}{\partial \xi^2}\right)\left[\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \xi}{\partial y}\right)^2\right] + \left(\frac{\partial^2 \phi}{\partial \eta^2}\right)\left[\left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2\right]$$

$$+2\,\frac{\partial^2 \phi}{\partial \xi \partial \eta}\left[\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \eta}{\partial x}\right) + \left(\frac{\partial \xi}{\partial y}\right)\left(\frac{\partial \eta}{\partial y}\right)\right]$$

$$+\frac{\partial \phi}{\partial \xi}\left[\left(\frac{\partial^2 \xi}{\partial x^2}\right) + \left(\frac{\partial^2 \xi}{\partial y^2}\right)\right] + \frac{\partial \phi}{\partial \eta}\left[\left(\frac{\partial^2 \eta}{\partial x^2}\right) + \left(\frac{\partial^2 \eta}{\partial y^2}\right)\right] = 0$$

(11)

The transformed Laplace's equation in the computational $(\xi, \eta)$ space.

## Metrics and Jacobians

In transformed space, we know that,

$$\left.\begin{aligned} \xi &= \xi(x, y, t) \\ \eta &= \eta(x, y, t) \\ \tau &= \tau(t) \end{aligned}\right\} \longrightarrow (1)$$

In many applications, the transformation may be more conveniently expressed as the inverse of eqn (1).

$$\left.\begin{aligned} x &= x(\xi, \eta, \tau) \\ y &= y(\xi, \eta, \tau) \\ t &= t(\tau) \end{aligned}\right\} \longrightarrow (2)$$

In eqn (2), $\xi$, $\eta$ and $\tau$ are the independent variables.

The metric terms of eqn (1) are $\dfrac{\partial \xi}{\partial x}, \dfrac{\partial \eta}{\partial y}, \ldots$ and for the inverse forms $\dfrac{\partial x}{\partial \xi}, \dfrac{\partial y}{\partial \eta}, \ldots$

Consider a dependent variable in the governing flow equations such as the $x$-component of velocity, $u$. The total differential of $u$ is given by

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy$$

Then,

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \xi}$$

and

$$\frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \eta}$$

The two unknowns $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ can be found using Crame's rule as follows.

$$\frac{\partial u}{\partial x} = \frac{\begin{vmatrix} \frac{\partial u}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial u}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{vmatrix}}{\begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{vmatrix}} ; \quad \frac{\partial u}{\partial y} = \frac{\begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial u}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial u}{\partial \eta} \end{vmatrix}}{\begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{vmatrix}}.$$

The denominator determinant is identified as the Jacobian determinant, denoted by

$$J = \frac{\partial(x,y)}{\partial(\xi,\eta)} = \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{vmatrix}.$$

Hence,

$$\frac{\partial u}{\partial x} = \frac{1}{J}\left[\left(\frac{\partial u}{\partial \xi}\right)\left(\frac{\partial y}{\partial \eta}\right) - \left(\frac{\partial u}{\partial \eta}\right)\left(\frac{\partial y}{\partial \xi}\right)\right]$$

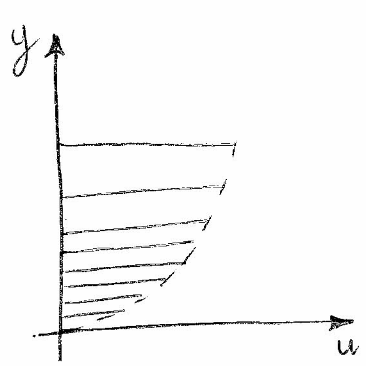
$$\frac{\partial u}{\partial y} = \frac{1}{J}\left[\left(\frac{\partial u}{\partial \eta}\right)\left(\frac{\partial x}{\partial \xi}\right) - \left(\frac{\partial u}{\partial \xi}\right)\left(\frac{\partial x}{\partial \eta}\right)\right]$$

These equations express the derivatives of the flow field variables in physical space in terms of the derivatives of the flow fields variables in computational space.

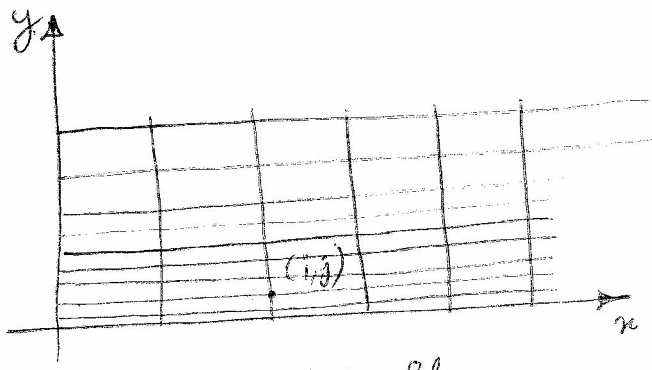
The transformed governing flow equations can be expressed in terms of these inverse metrics and the Jacobian, $J$.

## Coordinate Stretching.

It consists of stretching the grid in one or more coordinate directions. Assume that we are dealing with the viscous flow over a flat surface. The velocity profile in the physical plane is

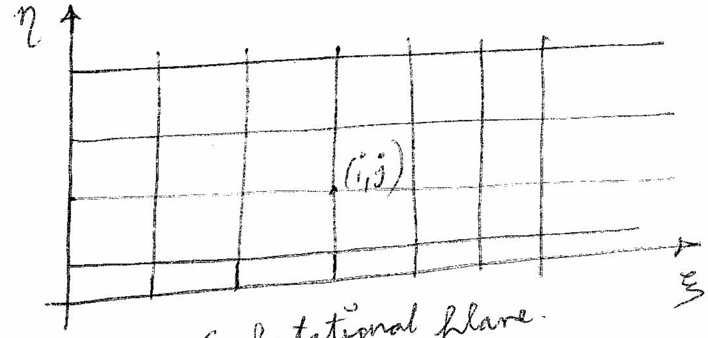



Velocity profile.




Physical plane

To calculate the details of this flow near the surface, a finely spaced grid in the $y$-direction should be used.




Computational plane.

A uniform grid is obtained in the computational plane by accomplishing the grid stretching. The transformation is

$$\xi = x$$
$$\eta = \ln(y+1)$$



Inverse transformation is

$$x = \xi$$
$$y = e^{\eta} - 1.$$

The inverse metrics are obtained as,

$$\frac{\partial x}{\partial \xi} = 1 \ ; \quad \frac{\partial x}{\partial \xi} = 0 \ ; \quad \frac{\partial y}{\partial \xi} = 0 \ ; \quad \frac{\partial y}{\partial \eta} = e^{\eta}$$

Let us consider the continuity equation for steady, two-dimensional flow.

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0$$

This equation can be formally transformed by means of the general results given by $\frac{\partial u}{\partial x}$ & $\frac{\partial u}{\partial y}$.

$$\frac{1}{J}\left[\frac{\partial(\rho u)}{\partial \xi}\left(\frac{\partial y}{\partial \eta}\right) - \frac{\partial(\rho u)}{\partial \eta}\left(\frac{\partial y}{\partial \xi}\right)\right] + \frac{1}{J}\left[\frac{\partial(\rho v)}{\partial \eta}\left(\frac{\partial x}{\partial \xi}\right) - \frac{\partial(\rho v)}{\partial \xi}\left(\frac{\partial x}{\partial \eta}\right)\right] = 0 \rightarrow (1)$$

Substituting inverse metrics in eqn (1) gives us.

$$e^{\eta}\frac{\partial(\rho u)}{\partial \xi} + \frac{\partial(\rho v)}{\partial \eta} = 0 \rightarrow (2)$$

Equation ② is the continuity equation in the computational plane. The metrics for direct transformation are given by,

$$\frac{\partial \xi}{\partial x} = 1 \; ; \quad \frac{\partial \xi}{\partial y} = 0 \; ; \quad \frac{\partial \eta}{\partial x} = 0 \; ; \quad \frac{\partial \eta}{\partial y} = \frac{1}{y+1}$$

Using the transformation given by $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, the continuity eqn can be written as,

$$\frac{\partial (\rho u)}{\partial \xi}\left(\frac{\partial \xi}{\partial x}\right) + \frac{\partial (\rho u)}{\partial \eta}\left(\frac{\partial \eta}{\partial x}\right) + \frac{\partial (\rho v)}{\partial \xi}\left(\frac{\partial \xi}{\partial y}\right) + \frac{\partial (\rho v)}{\partial \eta}\left(\frac{\partial \eta}{\partial y}\right) = 0 \quad \text{③}$$

Substituting the metric terms we have.

$$\frac{\partial (\rho u)}{\partial \xi} + \frac{1}{(y+1)}\left(\frac{\partial (\rho v)}{\partial \eta}\right) = 0$$

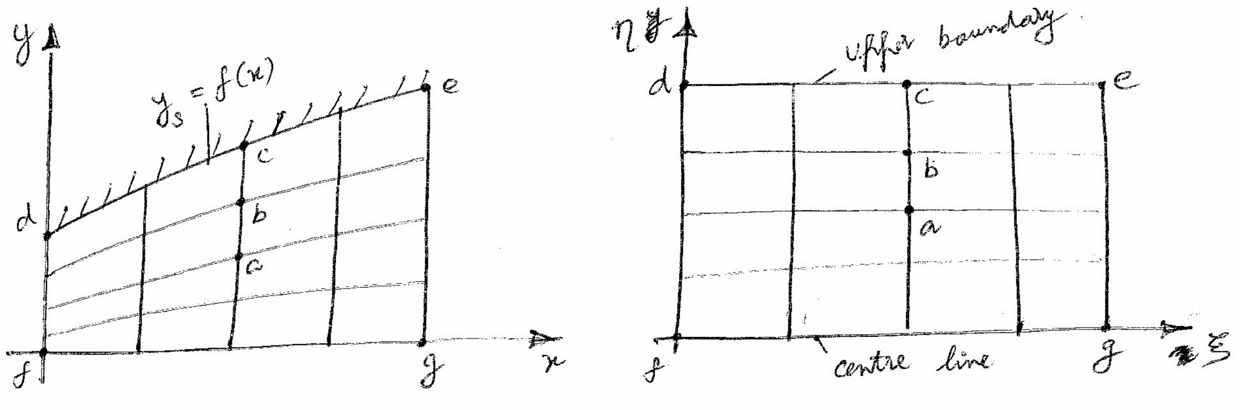
$$\frac{\partial (\rho u)}{\partial \xi} + \frac{1}{e^{\eta}} \frac{\partial (\rho v)}{\partial \eta} = 0$$

$$e^{\eta} \frac{\partial (\rho u)}{\partial \xi} + \frac{\partial (\rho v)}{\partial \eta} = 0 \longrightarrow \text{④}$$

The eqn ④ is identical to eqn ②. So, the transformed equation can be obtained from either the direct transformation or the inverse transformation.

## Boundary - Fitted Coordinate Systems.

Consider the flow through the divergent duct.




Physical plane      computational plane

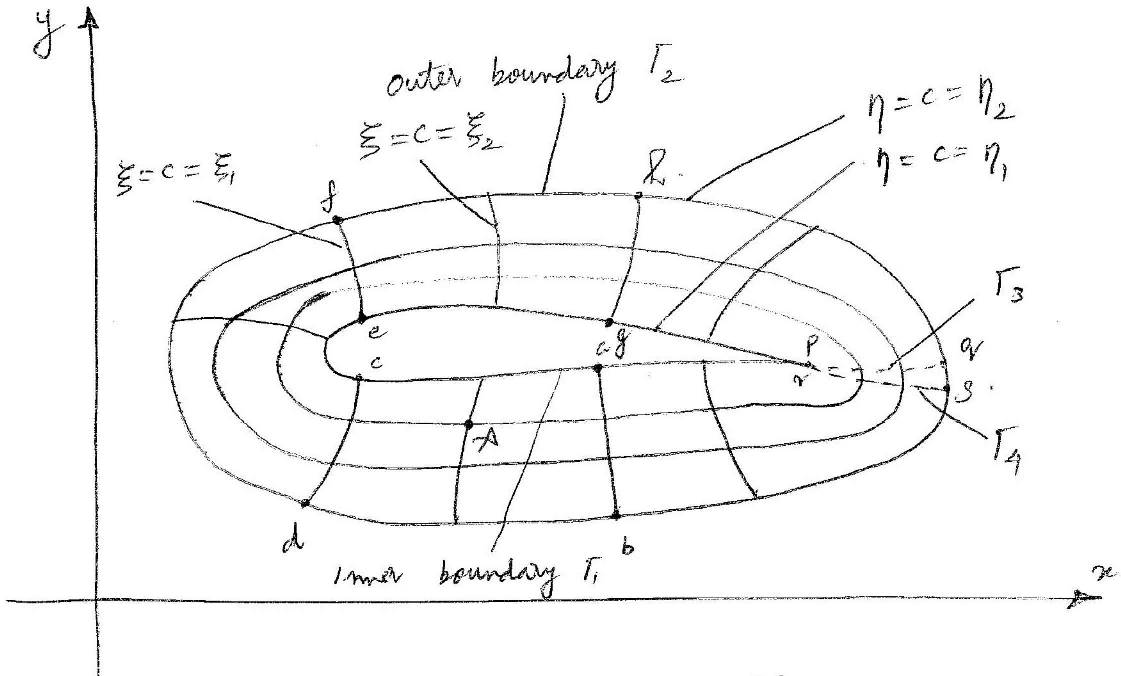
curve de is the upper wall of the duct and line fg is the centreline. Let draw the curvilinear grid in physical plane which allows both the upper boundary de and the centreline fg to be coordinate lines, exactly fitting these boundaries. In order to transform in computational plane let $y_s = f(x)$ be the ordinate of the upper surface de. Then the following transformation will result in a rectangular grid in $(\xi, \eta)$ space

$$\xi = x$$

$$\eta = y/y_s \qquad \text{where, } y_s = f(x)$$

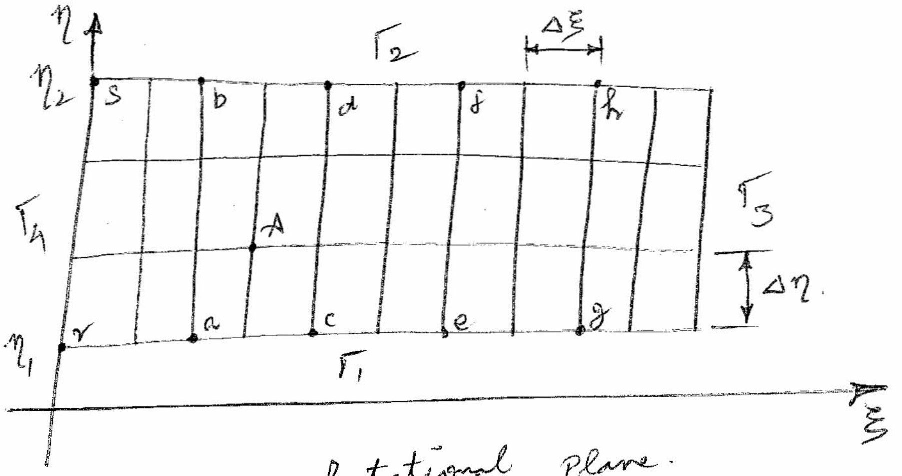
This is a simple example of a boundary - fitted coordinate system.

For more sophisticated example, let us consider the airfoil shape which is wrapped by curvilinear system around the airfoil, where one coordinate line $\eta = \eta_1 = c$ on the airfoil surface. This is the inner boundary of the grid, designated by $\Gamma_1$.




Airfoil in physical plane.

The outer boundary of the grid is labelled $\Gamma_2$ and is given by $\eta = \eta_2 = c$. Examining this grid, we see that it clearly fits the boundary, and hence it is a boundary-fitted coordinate system. The lines which fan out from the inner boundary $\Gamma_1$ and which intersect the outer boundary $\Gamma_2$ are lines of constant $\xi$, such as line ef for which $\xi = \xi_1 = c$.




Computational plane.

The boundary conditions are known everywhere along the boundary. Therefore, let us consider the transformation to be defined by an elliptic partial differential equation. One of the simplest elliptic equations is Laplace's equation,

$$\left.\begin{aligned} \frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} &= 0 \\ \frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} &= 0. \end{aligned}\right\} \longrightarrow (1)$$

Where we have Dirichlet boundary conditions,

$$\eta = \eta_1 = \text{constant on } \Gamma_1$$

$$\eta = \eta_2 = \text{constant on } \Gamma_2.$$

and $\xi = \xi(x,y)$ is specified on both $\Gamma_1$ and $\Gamma_2$.

Let us look more closely at the physical and computational planes. For order to construct a rectangular grid in the computational plane, a cut must be made in the physical plane at the trailing edge of the airfoil. This cut can be visualized as two lines superimposed on each other. The line pq denoted by $\Gamma_3$ represents a boundary line for the physical space above pq and the line rs denoted by $\Gamma_4$ represents a boundary line for the physical space below rs.

Let us emphasize that values of $(x, y)$ are known along all four boundaries, $\Gamma_1$, $\Gamma_2$, $\Gamma_3$ and $\Gamma_4$. The key aspect of the elliptic grid generation approach is that, with the given boundary conditions, eqn ① is solved for the $(x, y)$ values which apply to all the internal points. The equations to be solved are the inverse of eqn ①, that is equations obtained ~~from~~ by interchanging the dependent and independent variables.

The result is

$$\left.\begin{aligned} \alpha \frac{\partial^2 x}{\partial \xi^2} - 2\beta \frac{\partial^2 x}{\partial \xi \partial \eta} + \gamma \frac{\partial^2 x}{\partial \eta^2} &= 0 \\ \alpha \frac{\partial^2 y}{\partial \xi^2} - 2\beta \frac{\partial^2 y}{\partial \xi \partial \eta} + \gamma \frac{\partial^2 y}{\partial \eta^2} &= 0 \end{aligned}\right\} \rightarrow ②$$

where,

$$\alpha = \left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2$$

$$\beta = \left(\frac{\partial x}{\partial \xi}\right)\left(\frac{\partial x}{\partial \eta}\right) + \left(\frac{\partial y}{\partial \xi}\right)\left(\frac{\partial y}{\partial \eta}\right)$$

$$\gamma = \left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2$$

In eqn (2), $x$ & $y$ are expressed as the dependent variables. Eqn (2) is solved, along with the given boundary conditions for $(x, y)$ on $\Gamma_1$, $\Gamma_2$, $\Gamma_3$ and $\Gamma_4$, to obtain the values of $(x, y)$ which correspond to the uniformly spaced grid points in the computational $(\xi, \eta)$ plane. Thus, a given grid point $(\xi_i, \eta_i)$ in the computational plane corresponds to the calculated grid point $(x_i, y_i)$ in physical space.

## Adaptive Grids

An adaptive grid is a grid network that automatically clusters grid points in regions of high flow field gradients. It uses the solution of the flow field properties to locate the grid points in the physical plane. The adaptive grid evolves in steps of time in conjunction with a time-dependent solution of the governing flow field equations, which computes the flow field variables in steps of time. ~~[illegible]~~ The advantages are

i) increased accuracy for a fixed number of grid points.

(or)

ii) for a given accuracy, fewer grid points are needed.

Here, the transformation is expressed in the form.

$$\Delta x = \frac{B \Delta \xi}{1 + b \frac{\partial g}{\partial x}} \longrightarrow (1)$$

$$\Delta y = \frac{C \Delta \eta}{1 + c \frac{\partial g}{\partial y}} \longrightarrow (2)$$

In eqns (1) & (2), $\Delta \xi$ and $\Delta \eta$ are fixed, uniform grid spacings in the computational $(\xi, \eta)$ plane, b and c are constants chosen to increase or decrease the effect of the gradient in changing the grid spacing in the physical plane. B and C are scale factors and $\Delta x$ and $\Delta y$ are the new grid spacings in the physical plane. The governing flow equations are solved in the computational plane, where the x, y and t derivatives are transformed.

For this time-dependent adaptive grid,

$$\frac{\partial \xi}{\partial t} = \left(\frac{\partial \xi}{\partial t}\right)_{x,y} \quad ; \quad \frac{\partial \eta}{\partial t} = \left(\frac{\partial \eta}{\partial t}\right)_{x,y} .$$

are finite. When dealing with the time transformed flow equations in the computational plane, all three terms on the right hand side are finite, the time metrics $\frac{\partial \xi}{\partial t}$ and $\frac{\partial \eta}{\partial t}$ automatically take into account the movement of the adaptive grid during the solution of the governing flow equations.

The values of the time metrics in the form $\frac{\partial \xi}{\partial t}$, $\frac{\partial \eta}{\partial t}$ are a bit cumbersome to evaluate. But the related time metrics $\frac{\partial x}{\partial t}$ and $\frac{\partial y}{\partial t}$ are much easier to evaluate, because they come from

$$\left(\frac{\partial x}{\partial t}\right)_{\xi,\eta} = \frac{\Delta x}{\Delta t} \quad ; \quad \left(\frac{\partial y}{\partial t}\right)_{\xi,\eta} = \frac{\Delta y}{\Delta t} .$$

Let us find the relationship between these two sets of time metrics. Consider

$$x = x(\xi, \eta, \tau)$$

$$y = y(\xi, \eta, \tau).$$

92

Hence

$$dx = \left(\frac{\partial x}{\partial \xi}\right)_{\eta,\tau} d\xi + \left(\frac{\partial x}{\partial \eta}\right)_{\xi,\tau} d\eta + \left(\frac{\partial x}{\partial \tau}\right)_{\xi,\eta} d\tau$$

$$dy = \left(\frac{\partial y}{\partial \xi}\right)_{\eta,\tau} d\xi + \left(\frac{\partial y}{\partial \eta}\right)_{\xi,\tau} d\eta + \left(\frac{\partial y}{\partial \tau}\right)_{\xi,\eta} d\tau$$

Thus, from these results we write

$$\cancelto{0}{\left(\frac{\partial x}{\partial t}\right)_{x,y}} = \left(\frac{\partial x}{\partial \xi}\right)_{\eta,\tau}\left(\frac{\partial \xi}{\partial t}\right)_{x,y} + \left(\frac{\partial x}{\partial \eta}\right)_{\xi,\tau}\left(\frac{\partial \eta}{\partial t}\right)_{x,y} + \left(\frac{\partial x}{\partial \tau}\right)_{\xi,\eta}\cancelto{1}{\left(\frac{\partial \tau}{\partial t}\right)_{x,y}}.$$

$$-\left(\frac{\partial x}{\partial \tau}\right)_{\xi,\eta} = \left(\frac{\partial x}{\partial \xi}\right)_{\eta,\tau}\left(\frac{\partial \xi}{\partial t}\right)_{x,y} + \left(\frac{\partial x}{\partial \eta}\right)_{\xi,\tau}\left(\frac{\partial \eta}{\partial t}\right)_{x,y}. \rightarrow (3)$$

Similarly,

$$-\left(\frac{\partial y}{\partial \tau}\right)_{\xi,\eta} = \left(\frac{\partial y}{\partial \xi}\right)_{\eta,\tau}\left(\frac{\partial \xi}{\partial t}\right)_{x,y} + \left(\frac{\partial y}{\partial \eta}\right)_{\xi,\tau}\left(\frac{\partial \eta}{\partial t}\right)_{x,y}. \rightarrow (4)$$

Using Cramer's rule.

$$\left(\frac{\partial \xi}{\partial t}\right) = \frac{\begin{vmatrix} -\frac{\partial x}{\partial \tau} & \frac{\partial x}{\partial \eta} \\ -\frac{\partial y}{\partial \tau} & \frac{\partial y}{\partial \eta} \end{vmatrix}}{\begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{vmatrix}} ; \quad \frac{\partial \eta}{\partial t} = \frac{\begin{vmatrix} \frac{\partial x}{\partial \xi} & -\frac{\partial x}{\partial \tau} \\ \frac{\partial y}{\partial \xi} & -\frac{\partial y}{\partial \tau} \end{vmatrix}}{\begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{vmatrix}}$$

W.K.T $\tau = t$ and the denominator is the Jacobian J, then

$$\frac{\partial \xi}{\partial t} = \frac{1}{J}\left[-\left(\frac{\partial x}{\partial t}\right)\left(\frac{\partial y}{\partial \eta}\right) + \left(\frac{\partial y}{\partial t}\right)\left(\frac{\partial x}{\partial \eta}\right)\right]$$

$$\frac{\partial \eta}{\partial t} = \frac{1}{J}\left[-\left(\frac{\partial y}{\partial t}\right)\left(\frac{\partial x}{\partial \xi}\right) + \left(\frac{\partial x}{\partial t}\right)\left(\frac{\partial y}{\partial \xi}\right)\right]$$

For an adaptive grid, the governing flow equations, when transformed for solution in the computational $(\xi, \eta)$ plane, must contain all the terms in the time transformation.

## Delaunay Triangulation

The Delaunay triangulation provides a grid where a fixed set of rules applies to the construction, and the grid properties include the following.

i) Given a set of points, the triangulation is unique.

ii) The triangulation produces the most equilateral mesh for the given point set.

iii) The grid point generation and the triangulation are decoupled.

The Delaunay triangulation has a number of implementations and includes the diagonal swapping, the Bowyer insertion scheme and the sweepline method.

The disadvantages are.

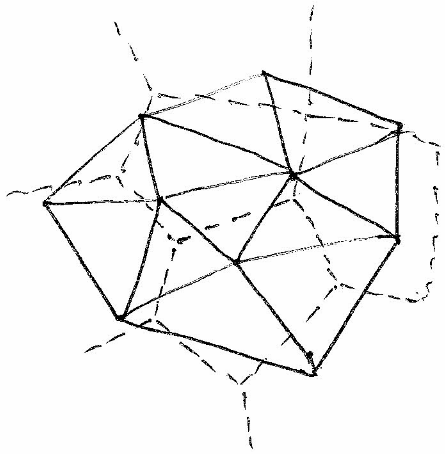
i) Lack of uniqueness when four points lie on a circle and the counterpart in 3-D.

ii) The complex logic required to preserve boundaries.

iii) The lack of uniqueness resulting from the numerical implementation of the analytical theory of the triangulation.

iv) The solution errors associated with high aspect ratio or elongated cells.

Given a point set $P = p_i(x_i)$ that is not colinear and does not have four points that lie on a circle, the set of points that is closer to vertex $v_i$ than any other vertex is called Voronoi Polygon. This is illustrated in the following figure, where the Voronoi polygons are shown for a finite set of points.



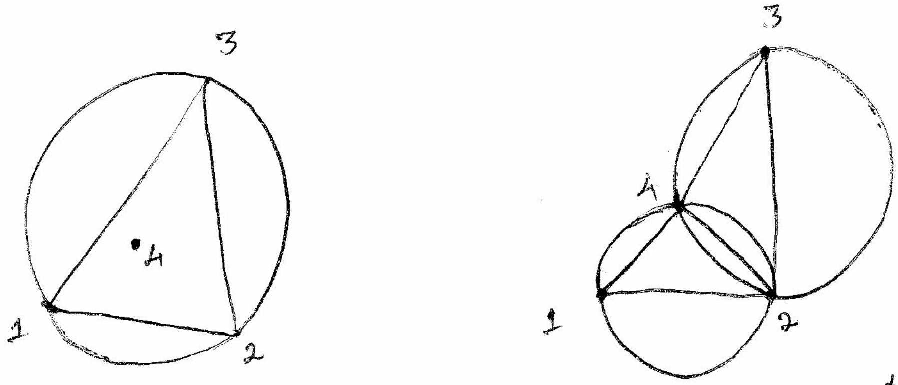


## Voronoi and Delaunay tesselations

The dashed lines are the voronoi polygons formed by constructing cells with sides corresponding to the perpendicular bisectors of the line segments in the triangulation. The vertices of the polygons are formed from the intersection of the perpendicular bisectors of the lines connecting the points, $P = P_i(x_i)$. As the mesh grows, more cells are added due to the addition of more lines segments connecting the points in the triangulation. The complete set of polygons including those closed on the interior and those open on the boundary of the domain is referred to as the Voronoi tesselation of the domain.

When the nuclei ($P_i$) of the voronoi polygons are connected to the two nearest neighbors, the resulting structure is called the Delaunay triangulation or Delaunay tesselation.

The circumcircle test is the simplest method to construct the Delaunay mesh and determine the connectivity of a set of points. For the planar case, three points determine a circle. For a triangular cell, the cell is a valid cell if no other points falls within the circle defined by the forming points of the circle. This is the standard test used to complete the connections for the Delaunay tesselation.




Incorrect connectivity

Correct connectivity

Circumcircle Test

The first connection formed by connecting points 1, 2 and 3 encloses point 4. This violates the circle criterion. The proper connections are made with two cells.